A FIXED POINT THEOREM FOR MAPPINGS WITH A NONEXPANSIVE ITERATE¹

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ABSTRACT. Let X be a reflexive Banach space which has strictly convex norm and suppose K is a nonempty, bounded, closed and convex subset of X . Suppose $T: K \rightarrow K$ has the property that, for some positive integer N , T^N is nonexpansive ($\|T^N x - T^N y\| \leq \|x - y\|$ for all $x, y \in K$). A function $\gamma(N)$ is determined, $\gamma(N) > 1$, such that if $\|T^j x - T^j y\| \leq k\|x - y\|$ for all $x, y \in K$, $1 \leq j \leq N-1$, where $k < \gamma(N)$, then T has a fixed point in K .

Let X be a Banach space and K a subset of X . A mapping $T: K \rightarrow X$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$. F. Browder [3], D. Göhde [5], and the author [6] proved independently that if X is a uniformly convex Banach space, K a nonempty, bounded, closed and convex subset of X , and T is a nonexpansive mapping of K into K , then T always has at least one fixed point. (The slightly weaker assumptions in [6] are that X be reflexive and that K possess the property Brodskiĭ and Mil'man [2] call "normal structure.") If one merely assumes that for some positive integer N the N th iterate, T^N , of T is nonexpansive then T need not have a fixed point since, in particular, a periodic homeomorphism of the unit ball of Hilbert space may be fixed point free (Klee [7]).

In this paper we respond to a question raised recently by K. Goebel in [4] and obtain conditions sufficient to guarantee existence of fixed points for mappings T such that T^N is nonexpansive. Our principal result is the following:

THEOREM 1. *Let X be a reflexive Banach space which has strictly convex norm and suppose K is a nonempty, bounded, closed and convex subset of X which possesses normal structure. Suppose the mapping $T: K \rightarrow K$ has the property that for some integer $N > 1$, T^N is nonexpansive, and suppose further that there is a constant k satisfying*

$$(i) \quad N^{-2}[(N-1)(N-2)k^2 + 2(N-1)k] < 1$$

such that $\|T^j x - T^j y\| \leq k\|x - y\|$ for all $x, y \in K$, $1 \leq j \leq N-1$. Then T has a fixed point in K .

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The concept of "normal structure" enters here only so that the theorem of [6] might apply. To say that K has *normal structure* means that every convex subset H of K which contains more than one point contains a point x such that $\sup\{\|x-y\|: y \in H\} < \text{diam } H$.

For the case $N=2$, condition (i) becomes $k < 2$. Goebel [4] has obtained a sharper result in this case using the concept of "modulus of convexity." Specifically, let B be the closed unit ball in X and let

$$\delta(\epsilon) = \inf[1 - \|x-y\|/2 : x, y \in B, \|x-y\| \geq \epsilon].$$

With X and K as in Theorem 1, results of Goebel [4] imply that a mapping $T: K \rightarrow K$ has a fixed point if $\|T^2x - T^2y\| \leq \|x-y\|$ and $\|Tx - Ty\| \leq k\|x-y\|$ where

$$(k/2)(1 - \delta(2/k)) < 1.$$

The above inequality holds for $k < 2$ in an arbitrary Banach space (and for $k < \sqrt{5}$, for example, in Hilbert space). This yields our version of Theorem 1 in the case $N=2$ in arbitrary spaces, and an improved version in special settings.

We obtain Theorem 1 as a consequence of the theorem of [6] and the following:

THEOREM 2. *Let K be a nonempty, closed, convex subset of a Banach space X and suppose $T: K \rightarrow K$ has the property that, for some integer $N > 1$, T^N is the identity mapping on K . Suppose further that there is a constant k satisfying condition (i) of Theorem 1 such that $\|T^jx - T^jy\| \leq k\|x-y\|$ for all $x, y \in K$, $1 \leq j \leq N-1$. Then the mapping T has a fixed point in K .*

Before proving the above theorems, we establish the following:

LEMMA. *Let K be a convex subset of a Banach space and suppose $T: K \rightarrow K$ has the property that, for some integer $N > 2$, T^N is the identity mapping on K . Suppose further that there is a constant $k < (N-1)/(N-2)$ such that $\|T^jx - T^jy\| \leq k\|x-y\|$ for $x, y \in K$, $1 \leq j \leq N-1$. If the mapping $G: K \rightarrow K$ is defined by*

$$Gx = N^{-1}(x + Tx + \cdots + T^{N-1}x), \quad x \in K,$$

then the mappings T and G have the same fixed points.

PROOF. Clearly $Tx = x$ implies $Gx = x$. Suppose, then, that $Gx = x$. From the definition of G we obtain

$$x = (N-1)^{-1}(Tx + T^2x + \cdots + T^{N-1}x).$$

This, with the Lipschitz condition on T^i yields, for $1 \leq i \leq N-1$,

$$\begin{aligned}
\|x - T^i x\| &= \|(N-1)^{-1}(Tx + T^2x + \cdots + T^{N-1}x) - T^i x\| \\
&\leq (N-1)^{-1}(\|Tx - T^i x\| + \cdots + \|T^{i-1}x - T^i x\| \\
&\quad + \|T^{i+1}x - T^i x\| + \cdots + \|T^{N-1}x - T^i x\|) \\
&\leq k(N-1)^{-1}(\|x - T^{i-1}x\| + \cdots + \|x - Tx\| \\
&\quad + \|T^{N-1}x - x\| + \cdots + \|T^{i+1}x - x\|).
\end{aligned}$$

Since $\|x - T^i x\|$ does not appear in the last summand,

$$\sum_{i=1}^{N-1} \|x - T^i x\| \leq k(N-1)^{-1}(N-2) \sum_{i=1}^{N-1} \|x - T^i x\|.$$

Because $k < (N-1)(N-2)^{-1}$, the above implies $\|x - Tx\| = 0$, i.e., $Tx = x$.

PROOF OF THEOREM 2. Define the mapping $G: K \rightarrow K$ by

$$Gx = N^{-1}(x + Tx + \cdots + T^{N-1}x), \quad x \in K.$$

For $j=0, 1, \dots$, let

$$\delta_j = \sup\{\|G^j x - T^i G^j x\| : i = 1, \dots, N-1\}.$$

Then

$$\begin{aligned}
\|G^j x - G^{j+1} x\| &= \|G^j x - N^{-1}(G^j x + TG^j x + \cdots + T^{N-1}G^j x)\| \\
(1) \quad &\leq N^{-1} \sum_{i=1}^{N-1} \|G^j x - T^i G^j x\| \leq (N-1)N^{-1}\delta_j.
\end{aligned}$$

Next we compare δ_j with δ_{j-1} . For $j=1, 2, \dots, 1 \leq i \leq N-1$,

$$\begin{aligned}
\|G^j x - T^i G^j x\| &= \|N^{-1}(G^{j-1}x + TG^{j-1}x + \cdots + T^{N-1}G^{j-1}x) - T^i G^j x\| \\
(2) \quad &\leq N^{-1} \sum_{p=0}^{N-1} \|T^p G^{j-1}x - T^i G^j x\|.
\end{aligned}$$

There are $N-1$ terms in the last summand of the form $\|T^p G^{j-1}x - T^i G^j x\|$, $p \neq i$. For fixed p ,

$$(3) \quad \|T^p G^{j-1}x - T^i G^j x\| \leq k\|T^s G^{j-1}x - G^j x\|$$

for an appropriate integer s , $1 \leq s \leq N-1$. (Note that if $p > i$ then $s = p - i$ while if $p < i$ then $s = N + p - i$.) Since

$$\begin{aligned}
&\|T^s G^{j-1}x - G^j x\| \\
(4) \quad &= \|T^s G^{j-1}x - N^{-1}(G^{j-1}x + TG^{j-1}x + \cdots + T^{N-1}G^{j-1}x)\| \\
&\leq N^{-1} \sum_{p=0}^{N-1} \|T^s G^{j-1}x - T^p G^{j-1}x\|
\end{aligned}$$

and since $\|T^s G^{j-1}x - T^p G^{j-1}x\| \leq k\delta_{j-1}$ we obtain from (4)

$$(5) \quad \|T^s G^{j-1}x - G^jx\| \leq (N-2)N^{-1}k\delta_{j-1} + N^{-1}\delta_{j-1}, \quad 1 \leq s \leq N-1.$$

Combining (5) with (2) and (3) yields

$$\begin{aligned} & \|G^jx - T^i G^jx\| \\ & \leq \frac{N-1}{N} \sup_{p \neq i} \|T^p G^{j-1}x - T^i G^jx\| + \frac{1}{N} \|T^i G^{j-1}x - T^i G^jx\| \\ & \leq \left(\frac{N-1}{N}\right) k \sup_{1 \leq s \leq N-1} \|T^s G^{j-1}x - G^jx\| + \frac{1}{N} k \|G^{j-1}x - G^jx\| \\ & \leq \frac{N-1}{N} k \left[\frac{N-2}{N} k\delta_{j-1} + \frac{1}{N} \delta_{j-1} \right] + \frac{1}{N} k \left(\frac{N-1}{N} \right) \delta_{j-1}. \end{aligned}$$

Therefore

$$\delta_j \leq N^{-2}[(N-1)(N-2)k^2 + 2(N-1)k]\delta_{j-1}.$$

Letting $\alpha = N^{-2}[(N-1)(N-2)k^2 + 2(N-1)k]$, we have, for $j=1, 2, \dots$,

$$\|G^jx - G^{j+1}x\| \leq (N-1)N^{-1}\delta_j \leq (N-1)N^{-1}\alpha\delta_{j-1} < \alpha^j\delta_0.$$

Since $\alpha < 1$, this implies $\{G^n x\}$ is a Cauchy sequence. Because G is continuous, the limit z of $\{G^n x\}$ is fixed under G . If $N > 2$ then $k < (N-1)(N-2)^{-1}$ and the Lemma implies z is also a fixed point of T . For $N=2$ it is clear that every fixed point of G is also a fixed point of T .

PROOF OF THEOREM 1. The theorem of Kirk [6] implies that T^N has a nonempty fixed point set C in K which, because of strict convexity of X , must be convex. Clearly C is closed, $T: C \rightarrow C$, and T^N is the identity on C . Thus the assumptions for Theorem 2 are satisfied for T on C yielding a fixed point for T in C .

REMARK. A periodic isometry of a Banach space into itself must have a fixed point (as Theorem 2 shows). Results of [1, Corollary 3] imply that such is also the case for a pointwise periodic isometry if the space is reflexive, but we do not know if reflexivity is essential for this.

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