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SUMS OF CUBES OF GAUSSIAN INTEGERS

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ABSTRACT. Every Gaussian integer is the sum of the cubes of four Gaussian integers.

It is well known that every rational integer can be expressed as the sum of the cubes of five rational integers, positive or negative. It has long been conjectured that four such cubes would suffice, but despite recent progress, the conjecture remains unproved. It is therefore of interest that we prove:

THEOREM. *Every Gaussian integer can be expressed as the sum of the cubes of four Gaussian integers.*

PROOF. It is clear that the Gaussian integer x can be so expressed if and only if each of $-x$, ix and $-ix$ can be. Accordingly, if $x \equiv 0 \pmod 3$ it suffices to consider the three cases $x \equiv 0, 3, 3+3i \pmod 6$, and if $x \not\equiv 0 \pmod 3$ the four cases $x \equiv 1, 2, 1+i, 1+2i \pmod{3+3i}$. The result then follows from the following identities:

$$6z=(z+1)^3+(z-1)^3-2z^3,$$

$$6z+3=z^3+(4-z)^3+(2z-5)^3+(4-2z)^3,$$

$$6z+3+3i=(iz)^3+(iz-1+i)^3-(iz+i)^3-(iz-1)^3,$$

$$\begin{aligned}3(1+i)z+1=&-(z+1-i)^3+\{(2i-1)z+i-3\}^3\\&+\{(1-3i)z+3-2i\}^3-\{(2+2i)z+1+3i\}^3,\end{aligned}$$

$$3(1+i)z+2=(z+1)^3+(iz+1)^3-(iz)^3-z^3,$$

$$\begin{aligned}3(1+i)z+1+i=&\{(1+2i)z+24+12i\}^3+(2z+18-15i)^3\\&+(z+12-10i)^3+\{(i-1)z+18i-5\}^3,\end{aligned}$$

$$3(1+i)z+1+2i=(z-i)^3+\{(1+i)z+1\}^3-\{(1+i)z\}^3-(z+i)^3.$$

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