PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 29, No. 2, July 1971

MINIMAL HYPERSURFACES IN AN m-SPHERE

BANG-YEN CHEN

ABSTRACT. (1) A submanifold M^n of a euclidean space E^{n+2} of codimension 2 is a pseudo-umbilical submanifold with constant mean curvature if and only if it is a minimal hypersurface of a hypersphere of E^{n+2}. (2) A complete oriented minimal surface M^2 of a 3-sphere S^3 on which the Gauss curvature does not change its sign is either an equatorial sphere or a Clifford flat torus.

1. **Introduction.** Let $x: M^n \to R^m$ be an isometric immersion of a Riemannian manifold M^n of dimension n into an oriented Riemannian manifold R^m of dimension m $(m > n)$. For a unit normal vector $\mathbf{e}$ at $x(p)$, $p \in M^n$, there corresponds a selfadjoint transformation $A(\mathbf{e})$ of the tangent space $T_p(M^n)$ at p into itself, called the second fundamental form at $\mathbf{e}$. If $\mathbf{e}_{n+1}, \cdots, \mathbf{e}_m$ is an orthonormal basis of the normal space of M^n in R^m at $x(p)$, then the *mean curvature vector* $\boldsymbol{H}$ is given by

$$H = (1/n) \sum_{r=n+1}^{m} (\text{trace } A(\mathbf{e}_r))\mathbf{e}_r. \tag{1}$$

It is easy to verify that $\boldsymbol{H}$ is independent of the choice of the orthonormal basis $\mathbf{e}_{n+1}, \cdots, \mathbf{e}_m$. The length of the mean curvature vector $\boldsymbol{H}$ is called the *mean curvature.* If the mean curvature vector $\boldsymbol{H}=0$ identically, then the immersion $x: M^n \to R^m$ is called a *minimal immersion* and M^n is called a *minimal submanifold* of R^m. If the mean curvature vector $\boldsymbol{H}$ is nowhere zero and the second fundamental form at the direction of the mean curvature vector is proportional to the identity transformation of the tangent space of M^n everywhere, then the immersion $x: M^n \to R^m$ is called a *pseudo-umbilical immersion* and M^n is called a *pseudo-umbilical submanifold* of R^m.

In this paper we prove the following theorems:

THEOREM 1. *Let $x: M^n \to E^{n+2}$ be an isometric immersion of a Riemannian manifold M^n of dimension n into a euclidean space E^{n+2} of dimension $n+2$. Then M^n is a pseudo-umbilical submanifold of E^{n+2} with constant mean curvature if and only if M^n is a minimal hypersurface of a hypersphere of E^{n+2}.*

Received by the editors September 25, 1970.

AMS 1970 *subject classifications.* Primary 53A10, 53C40; Secondary 53B25, 53A05.

Key words and phrases. Second fundamental form, mean curvature vector, mean curvature, pseudo-umbilical submanifold, minimal hypersurface, equatorial sphere, Clifford flat torus, Gauss curvature.

THEOREM 2. *Let $x: M^2 \to S^3$ be a minimal immersion of a complete oriented surface M^2 into a 3-sphere S^3. If the Gauss curvature K of M^2 does not change its sign, then M^2 is immersed as an equatorial sphere or a Clifford flat torus.*

REMARK 1. In the case of minimal surfaces in a 3-sphere S^3 of constant Gauss curvature, Lawson proved the following local rigidity theorem [**3**]: If M^2 is a minimal surface in S^3 of constant Gauss curvature, then either M^2 is totally geodesic or M^2 is an open piece of the Clifford flat torus.

2. **Preliminaries.** Let $x: M^n \to E^{n+2}$ be an isometric immersion of a Riemannian manifold M^n of dimension n into a euclidean space E^{n+2} of dimension $n+2$. Let $F(M^n)$ and $F(E^{n+2})$ be the bundles of orthonormal frames of M^n and E^{n+2} respectively. Let B be the set of elements $b = (p, e_1, \cdots, e_n, e_{n+1}, e_{n+2})$ such that $(p, e_1, \cdots, e_n) \in F(M^n)$ and $(x(p), e_1, \cdots, e_{n+2}) \in F(E^{n+2})$ whose orientation is coherent with that of E^{n+2}, identifying e_i with $dx(e_i)$, $i = 1, \cdots, n$. Define $\tilde{x}: B \to F(E^{n+2})$ by $\tilde{x}(b) = (x(p), e_1, \cdots, e_{n+2})$.

The structure equations of E^{n+2} are given by

$$
\begin{aligned}
&dx = \sum_A \omega'_A e_A, \qquad de_A = \sum_B \omega'_{AB} e_B, \qquad \omega'_{AB} + \omega'_{BA} = 0, \\
(2) \quad &d\omega'_A = \sum_B \omega'_B \wedge \omega'_{BA}, \qquad d\omega'_{AB} = \sum_C \omega'_{AC} \wedge \omega'_{CB}, \\
&\qquad\qquad\qquad A, B, C, \cdots = 1, 2, \cdots, n+2,
\end{aligned}
$$

where ω'_A, ω'_{AB} are differential 1-forms on $F(E^{n+2})$. Let ω_A, ω_{AB} be the induced 1-forms on B from ω'_A, ω'_{AB} by the mapping $\tilde{x}$. Then we have

$$\omega_r = 0, \qquad r, t, \cdots = n+1, n+2.$$

Hence, from (2), we get

$$\sum_i \omega_i \wedge \omega_{ir} = 0, \qquad i, j, k, \cdots = 1, \cdots, n.$$

From this and a lemma of Cartan, we can write

$$\omega_{ir} = \sum_j A_{rij} \omega_j, \qquad A_{tij} = A_{tji}. \tag{3}$$

Moreover, from (2), we get

$$d\omega_i = \sum_j \omega_j \wedge \omega_{ji}, \qquad d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB}. \tag{4}$$

For each unit normal vector $\boldsymbol{e}$ at $x(p)$, if we put $\boldsymbol{e}=(\cos\theta)\boldsymbol{e}_{n+1}+(\sin\theta)\boldsymbol{e}_{n+2}$, then the second fundamental form $A(\boldsymbol{e})$ at $\boldsymbol{e}$ is the linear transformation given by

$$(A(\mathbf{e}))(\boldsymbol{e}_i) = \sum_j ((\cos\theta)A_{n+1ij} + (\sin\theta)A_{n+2ij})\boldsymbol{e}_j, \qquad i = 1, 2, \cdots, n. \tag{5}$$

3. **Proof of Theorem** 1. Suppose that the immersion $x: M^n \to E^{n+2}$ is a pseudo-umbilical immersion with constant mean curvature α. Then, by the definition, the mean curvature vector $\boldsymbol{H}$ is nowhere zero. Hence we can choose a unit normal vector $\bar{\boldsymbol{e}}_{n+1}$ in the direction of $\boldsymbol{H}$, that is $\boldsymbol{H}=\alpha\bar{\boldsymbol{e}}_{n+1}$. Therefore, we can suitably choose a local cross section of $M^n \to B$, say $(p, \bar{\boldsymbol{e}}_1, \cdots, \bar{\boldsymbol{e}}_n, \bar{\boldsymbol{e}}_{n+1}, \bar{\boldsymbol{e}}_{n+2})$, such that the corresponding 1-forms $\bar{\omega}_i$, $\bar{\omega}_{AB}$ of ω_i, ω_{AB} with respect to this local cross section satisfy the following relations:

$$\bar{\omega}_{in+2} = f_i\bar{\omega}_i, \qquad i = 1, 2, \cdots, n, \qquad f_1 + f_2 + \cdots + f_n = 0. \tag{6}$$

LEMMA 1. *Let $x: M^n \to E^{n+2}$ be a pseudo-umbilical immersion of M^n into E^{n+2}. Then the mean curvature α is constant if and only if the form $\bar{\omega}_{n+1,n+2}$ vanishes identically.*

PROOF. Since the immersion x is a pseudo-umbilical immersion and $\boldsymbol{H}=\alpha\bar{\boldsymbol{e}}_{n+1}$, we have

$$\bar{\omega}_{in+1} = \alpha\bar{\omega}_i, \qquad i = 1, 2, \cdots, n. \tag{7}$$

Hence, if the form $\bar{\omega}_{n+1,n+2}=0$ identically, then by using (4) and (6), we have

$$d\alpha \wedge \bar{\omega}_i = \bar{\omega}_{i,n+2} \wedge \bar{\omega}_{n+2,n+1} = 0, \qquad i = 1, 2, \cdots, n, \tag{8}$$

which imply that the mean curvature α is constant.

Conversely, if the mean curvature α is constant, then, by (3) and (8), we can easily prove that

$$\bar{\omega}_{i,n+2} = 0, \qquad i = 1, 2, \cdots, n, \tag{9}$$

on the open subset $U=\{p\in M^n;\ \bar{\omega}_{n+1,n+2}\neq 0 \text{ at } p\}$. By taking the exterior differentiation of (9) and applying (4), we can easily prove that

$$\bar{\omega}_i \wedge \bar{\omega}_{n+1,n+2} = 0, \qquad i = 1, 2, \cdots, n, \tag{10}$$

on the open subset U. This implies that $\bar{\omega}_{n+1,n+2}=0$ on U. Therefore we get $U=\varnothing$. This completes the proof of the lemma.

LEMMA 2. *If $x: M^n \to E^{n+2}$ is a pseudo-umbilical immersion and the* *mean curvature α is constant, then M^n is immersed in a hypersphere of E^{n+2}.*

PROOF. Consider the mapping $y: M^n \rightarrow E^{n+2}$ defined by $y(p)=x(p)+(1/\alpha)\bar{e}_{n+1}$, where $H=\alpha\bar{e}_{n+1}$. Then, taking account of $\bar{\omega}_{n+1,n+2}=0$ which is a direct consequence of Lemma 1, we have $dy(p)=0$. This means that M^n is immersed in a hypersphere of E^{n+2}.

LEMMA 3. *Let $x: M^n \rightarrow E^{n+2}$ be an isometric immersion of M^n into E^{n+2} such that M^n is immersed as a minimal submanifold of a hypersphere of E^{n+2}. Then M^n is a pseudo-umbilical submanifold of E^{n+2} with constant mean curvature.*

PROOF. Without loss of generality, we can assume that M^n is immersed as a minimal submanifold of the unit hypersphere of E^{n+2} centered at the origin. In this case, the position vector field X is a unit normal vector field of M^n in E^{n+2}. Since M^n is a minimal hypersurface of the unit hypersphere of E^{n+2} centered at the origin, we can easily prove that the mean curvature vector H of M^n in E^{n+2} is parallel to the position vector field X. By choosing the cross section $(p, \bar{e}_1, \cdots, \bar{e}_n, \bar{e}_{n+1}, \bar{e}_{n+2})$ of $M^n \rightarrow B$ with $\bar{e}_{n+1}=X$, we have

$$(11) \quad \bar{A}_{n+1ij} = -\delta_{ij}, \quad \bar{\omega}_{n+1,i} = -\sum \bar{A}_{n+1ij}\omega_j, \qquad i,j = 1, \cdots, n,$$

and

$$(12) \qquad \bar{\omega}_{n+1,n+2} = 0.$$

By (11), we know that the mean curvature is nowhere zero and M^n is a pseudo-umbilical submanifold of E^{n+2}. Moreover, by (12) and Lemma 2, we know that M^n has constant mean curvature. This completes the proof of the lemma.

Now, we return to the proof of the theorem:

Suppose that M^n is a pseudo-umbilical submanifold of E^{n+2} with constant mean curvature. By Lemma 2, without loss of generality, we can assume that M^n is immersed in the unit hypersphere centered at the origin. By taking a local cross section $(p, e_1, \cdots, e_{n+2})$ of $M^n \rightarrow B$ such that $e_{n+1}=X$, and $e_1, \cdots, e_n$ diagonalize the second fundamental form at e_{n+2}, we have

$$(13) \quad A(e_{n+1}) = \text{identity} \quad \text{and} \quad (A(e_{n+2}))(e_i) = h_i e_i, \quad i = 1, \cdots, n,$$

where h_i, $i=1, \cdots, n$, are functions on M^n. By (13), we know that the mean curvature vector H is given by

$$(14) \qquad H = e_{n+1} + (1/n)(h_1 + \cdots + h_n)e_{n+2}.$$

By the assumption that the mean curvature α is constant, we have

$$h_1 + \cdots + h_n = \text{constant}. \tag{15}$$

Hence, by (5), (13) and (14), we get

$$(A(\boldsymbol{H}/\alpha))(\mathbf{e}_i) = (1/n\alpha)((h_1 + \cdots + h_n)h_i + n)\mathbf{e}_i, \qquad i = 1, 2, \cdots, n. \tag{16}$$

Therefore, by the assumption of pseudo-umbilical, we have

$$\left(\sum_j h_j\right)h_1 = \left(\sum_j h_j\right)h_2 = \cdots = \left(\sum_j h_j\right)h_n. \tag{17}$$

If $h_1 + \cdots + h_n \neq 0$, then, by (17), we get $h_1 = h_2 = \cdots = h_n =$ constant on M^n. This shows that the immersion $x: M^n \to E^{n+2}$ is totally umbilical, i.e. the second fundamental form has the same eigenvalues for every normal direction. Thus, we know that M^n is immersed into a hypersphere of a hyperplane of E^{n+2} (see, for instance, [**1**]). If $h_1 + \cdots + h_n = 0$, then M^n is immersed as a minimal hypersurface in the unit hypersphere of E^{n+2}. In both cases, M^n is immersed as a minimal hypersurface of a hypersphere of E^{n+2}. The converse of this has been proved in Lemma 3. This completes the proof of the theorem.

REMARK 2. Lemma 1 and Lemma 2 have been proved in [**2**] for $n=2$.

4. Proof of Theorem 2. Suppose that $x: M^2 \to S^3$ be a minimal immersion of a complete oriented surface M^2 into a 3-sphere. Without loss of generality, we can regard S^3 as a hypersphere of E^4. By Lemma 3, we know that the immersion $x: M^2 \to S^3 \subset E^4$ is a pseudo-umbilical immersion in E^4 with constant mean curvature. Hence, by the assumption that the Gauss curvature K does not change its sign, we know that M^2 is immersed either as a sphere in a hyperplane of E^4 or as a Clifford flat torus [**2**]. Hence, by the fact that x is a minimal immersion of M^2 into S^3, we know that M^2 is either immersed as an equatorial sphere or immersed as a Clifford flat torus. This completes the proof of the theorem.

COROLLARY. *If M^2 is an oriented closed surface of genus $g \geqq 2$ with the Gauss curvature $K \leqq 0$, then M^2 cannot isometrically be immersed in a 3-sphere as a minimal submanifold.*

This corollary follows immediately from Theorem 2.

The author would like to express his thanks to Professors K. Yano, G. D. Ludden and D. E. Blair for the valuable conversations.

REFERENCES

1. B.-y. Chen, *On a theorem of Fenchel-Borsuk-Willmore-Chern-Lashof* (to appear).

2. T. Itoh, *Complete surfaces in E^4 with constant mean curvature*, Kōdai Math. Sem. Rep. **22** (1970), 150–158.

3. H. B. Lawson, Jr., *Local rigidity theorems for minimal hypersurfaces*, Ann. of Math. (2) **89** (1969), 187–197. MR **38** #6505.

4. R. Osserman, *Minimal varieties*, Bull. Amer. Math. Soc. **75** (1969), 1092–1120.

MICHIGAN STATE UNIVERSITY, EAST LANSING, MICHIGAN 48823