ON FUNCTIONS OF BOUNDED BOUNDARY ROTATION

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ABSTRACT. Let $U = \{z = re^{i\theta} \mid r < 1\}$. For $k \geq 2$ let V_k be the class of normalized analytic functions $f(z)$ such that the boundary rotation of $f(U)$ is at most $k\pi$. Let $A(r)$ be the integral

$$\int_0^{2\pi} \int_0^r |f'(pe^{i\theta})|^2 p dp d\theta,$$

$L(r)$ the length of the image of the circle $|z| = r$ under the mapping $f(z)$. In this paper the author proves that for $z \in U$ if $f(z) \in V_k$ then

$$\limsup_{r \rightarrow 1} \left(\sup_{f \in V_k} L(r) \right) \left(\pi A(r) \log \left(\frac{1+r}{1-r} \right) \right)^{-1/2} \leq k.$$

This generalizes to arbitrary $k \geq 2$ the recent result of Nunokawa for the case $k = 2$.

Let U be the unit disk, $|z| < 1$. For fixed $k \geq 2$, by V_k we denote the class of functions $f(z)$ satisfying the following conditions:

- (i) $f(z)$ is analytic and $f'(z) \neq 0$ for $z \in U$.
- (ii) $f(z)$ is normalized by the requirements $f(0) = 0$ and $f'(0) = 1$.
- (iii) The boundary rotation of $f(U)$ is at most $k\pi$, that is

$$(1) \quad \int_0^{2\pi} \left| \operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \right| d\theta \leq k\pi,$$

where $re^{i\theta} = z \in U$.

The concept of bounded boundary rotation originates from Löwner [1]. Extensive investigations are due to Paatero [3].

Let K denote the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, convex in U , i.e. $\operatorname{Re} \{ 1 + zf''(z)/f'(z) \} > 0$. It is clear that $V_2 = K$.

Let $C(r)$ denote the image of the circle $|z| = r < 1$ under some mapping $f(z)$, $L(r)$ the length of $C(r)$. By $A(r)$ we denote the integral

$$\int_0^{2\pi} \int_0^r |f'(t)|^2 p dp d\theta \quad (t = pe^{i\theta})$$

which is the generalized area of the image of the set $|z| \leq r$ under the mapping $f(z)$.

Recently, Nunokawa [2, p. 332] obtained:

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THEOREM N. If $f(z) \in K (= V_2)$ then

$$(2) \quad L(r) = O\left(A(r) \log \frac{1}{1-r}\right)^{1/2},$$

as $r \rightarrow 1$.

In this paper we shall extend Theorem N to all classes V_k by proving

THEOREM. Let $2 \leq k < \infty$. If $f(z) \in V_k$, then

$$(3) \quad \limsup_{r \rightarrow 1} \left(\sup_{f \in V_k} L(r) \right) \left(\pi A(r) \log \left(\frac{1+r}{1-r} \right) \right)^{-1/2} \leq k.$$

PROOF. Let a be a fixed constant, $0 < a < r = |z| < 1$. Nunokawa [2, p. 333] has shown that if $f(z)$ has a nonvanishing derivative $f'(z)$ in U then

$$L(r) = \int_0^{2\pi} |zf'(z)| d\theta \leq C + (A(r)J/a)^{1/2} \quad (z = re^{i\theta}),$$

where

$$C = \int_0^a \int_0^{2\pi} |tf''(t) + f'(t)| d\theta d\rho \quad (t = \rho e^{i\theta})$$

and

$$J = \int_0^r \int_0^{2\pi} \left| 1 + \frac{tf''(t)}{f'(t)} \right|^2 d\theta d\rho.$$

We observe that since the functions $f(z)$ of the class V_k comprise a normal family there exists a constant $A = A(a, k)$ depending only upon a and k such that $C \leq A$. An estimate for J for the class V_k follows from the following result due to Robertson [4, p. 1480]:

$$\begin{aligned} \int_0^{2\pi} \left| 1 + t \frac{f''(t)}{f'(t)} \right|^2 d\theta &= 2\pi + \rho^2 \int_0^{2\pi} \left| \frac{f''(t)}{f'(t)} \right|^2 d\theta \\ &\leq 2\pi \left(1 + \frac{k^2 \rho^2}{(1 - \rho^2)} \right) < \frac{2\pi k^2}{1 - \rho^2}. \end{aligned}$$

Then

$$J < 2\pi k^2 \int_0^r \frac{d\rho}{1 - \rho^2} = \pi k^2 \log \left(\frac{1+r}{1-r} \right).$$

Since $\int_0^{2\pi} |f'(z)|^2 d\theta \geq 2\pi$ it follows that $A(r) \log ((1+r)/(1-r))$ is unbounded as $r \rightarrow 1$. Since a can be chosen arbitrarily close to one the inequality (3) follows at once.

REMARK. It is interesting to note that recently, M. S. Robertson [5, p. 97] has considered a family of functions that includes the class K as a proper subfamily. For $-\frac{1}{2}\pi < \alpha < \frac{1}{2}\pi$, we say that $f(z) \in S_\alpha$ if

(i) $f(z)$ is analytic and $f'(z) \neq 0$ for $z \in U$.

(ii) $f(0) = 0$ and $f'(0) = 1$.

(iii) For $z \in U$,

$$\operatorname{Re} \left\{ e^{i\alpha} \left(1 + \frac{zf''(z)}{f'(z)} \right) \right\} > 0.$$

Clearly, $S_0 = K$. We can extend inequality (2) from the class K to all classes S_α ($|\alpha| < \frac{1}{2}\pi$) as follows:

For $f(z) \in S_\alpha$ ($|\alpha| < \frac{1}{2}\pi$) we have

$$(4) \quad \limsup_{r \rightarrow 1} \left(\sup_{f \in S_\alpha} L(r) \right) \left(\pi A(r) \log \left(\frac{1+r}{1-r} \right) \right)^{-1/2} \leq 2 \cos \alpha.$$

The proof of (4) is similar to that in our Theorem. Let $f(z) \in S_\alpha$. Then there is a function $P(z)$ with positive real part and regular in U , $P(0) = 1$, $|P^{(n)}(0)| \leq 2(n!)$, $n = 1, 2, 3, \dots$, such that

$$e^{i\alpha} \left(1 + \frac{zf''(z)}{f'(z)} \right) = P(z) \cos \alpha + i \sin \alpha.$$

Then

$$\begin{aligned} \int_0^{2\pi} \left| 1 + \frac{zf''(z)}{f'(z)} \right|^2 d\theta &= 2\pi + \int_0^{2\pi} \left| \frac{zf''(z)}{f'(z)} \right|^2 d\theta \\ &= 2\pi + \int_0^{2\pi} | (P(z) - 1) \cos \alpha |^2 d\theta \\ &\leq 2\pi + \frac{8\pi r^2 \cos^2 \alpha}{1 - r^2} = \frac{8\pi \cos^2 \alpha}{1 - r^2} (1 + o(1)) \end{aligned}$$

as $r \rightarrow 1$. Then (4) follows.

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