

INDUCED MATROIDS

R. A. BRUALDI¹

ABSTRACT. There are several known results concerning how matroids can be induced from given matroids by a bipartite graph and the properties that are inherited in this way. The purpose of this note is to extend some of these results to the situation where the bipartite graph is replaced by an arbitrary directed graph. We show how a directed graph and a matroid can be used to induce a new matroid. If the initial matroid is strongly base orderable, we prove that the induced matroid is also. In particular, a matroid induced from a free matroid by a directed graph is strongly base orderable. A consequence is that the cycle matroid of the complete graph on four nodes cannot be induced from a free matroid by any directed graph.

1. Let Γ be a directed graph in which each edge joins two distinct nodes. A *finite linear path* θ is a finite sequence (x_1, x_2, \dots, x_n) of distinct nodes where $n \geq 1$ and (x_i, x_{i+1}) is an edge of Γ ($1 \leq i < n$). Notice that for each node x , (x) is a finite linear path which we refer to as degenerate. The *initial node* of the path θ is x_1 , denoted by $\text{In } \theta$, while the *terminal node* of θ is x_n , denoted by $\text{Ter } \theta$; the set of nodes of θ is $\text{Nod } \theta = \{x_1, x_2, \dots, x_n\}$. If Θ is a collection of paths in Γ , then $\text{In } \Theta = \{\text{In } \theta : \theta \in \Theta\}$, $\text{Ter } \Theta = \{\text{Ter } \theta : \theta \in \Theta\}$, and $\text{Nod } \Theta = \{\text{Nod } \theta : \theta \in \Theta\}$. We say Θ is a set of pairwise node disjoint paths if $\theta_1, \theta_2 \in \Theta, \theta_1 \neq \theta_2$ imply $\text{Nod } \theta_1 \cap \text{Nod } \theta_2 = \emptyset$.

We formulate here the rule used in [3] for obtaining a theorem about pairwise node disjoint paths in a directed graph from the corresponding theorem for bipartite graphs. (Alternately, one can use the second approach of [3] to obtain directly the same result.) This rule can be formulated for infinite sets of paths as in [3], but we shall restrict our attention to finite sets of paths, although the directed graph Γ may be infinite. We shall then use this rule to prove some theorems about matroids.

2. Let Θ, Φ be two finite sets of pairwise node disjoint finite linear paths in the directed graph Γ . Let each node of $\text{Nod } \Theta \cap \text{Nod } \Phi$ be called an intersection node. The intersection nodes divide the paths of

Received by the editors October 19, 1970.

AMS 1970 subject classifications. Primary 05B35, 05C20; Secondary 05A05.

Key words and phrases. Matroid, directed graph, induced matroid, free matroid, cycle matroid, bipartite graph, pairwise node disjoint paths, base orderable matroid, strongly base orderable matroid.

¹ Research supported by NSF grant GP-17815.

Copyright © 1971, American Mathematical Society

Θ , resp. Φ , into a collection $\Sigma\Theta$, resp. $\Sigma\Phi$, of subpaths as follows. Let $\theta = (x_1, x_2, \dots, x_n) \in \Theta$. If no node of θ is an intersection node, then $\theta \in \Sigma\Theta$. Otherwise let $x_{k_1}, x_{k_2}, \dots, x_{k_t}$ ($1 \leq k_1 < k_2 < \dots < k_t \leq n$) be the nodes of θ which are intersection nodes. We then put (x_1, \dots, x_{k_1}) , $(x_{k_1}, \dots, x_{k_2})$, \dots , (x_{k_t}, \dots, x_n) in $\Sigma\Theta$. Notice that it may happen that $x_{k_1} = x_1$ so that (x_1, \dots, x_{k_1}) is the degenerate path (x_1) ; likewise it may happen that (x_{k_t}, \dots, x_n) is the degenerate path (x_n) . The set $\Sigma\Phi$ is constructed in an analogous way.

Let $\text{In } \Theta \cup \text{In } \Phi \subseteq X$ and $\text{Ter } \Theta \cup \text{Ter } \Phi \subseteq Y$. There is then associated a bipartite graph, denoted by $G(\Theta, \Phi)$ in the following way. Let X_I, Y_T, X', X'' be pairwise node disjoint sets such that there are bijections $x \rightarrow x_I$ of X to X_I , $y \rightarrow y_T$ of Y to Y_T , and $x \rightarrow x'$ and $x \rightarrow x''$ of $\text{Nod } \Theta \cap \text{Nod } \Phi$ to X' and X'' , respectively. The sets $X_I \cup X'$ and $Y_T \cup X''$ (which are disjoint by our assumptions) comprise the nodes of $G(\Theta, \Phi)$. We now construct the edges. Corresponding to each path in $\Sigma(\Theta)$ there is to be an edge of $G(\Theta, \Phi)$. Let $\theta = (x = x_1, x_2, \dots, x_n = y) \in \Theta$. If $\theta \in \Sigma(\Theta)$ we associate the edge (x_I, y_T) . Otherwise if (x, \dots, x_{k_1}) , $(x_{k_1}, \dots, x_{k_2})$, \dots , (x_{k_t}, \dots, y) are the subpaths of θ in $\Sigma(\Theta)$ we associate the edges (x_I, x'_{k_1}) , (x'_{k_1}, x'_{k_2}) , \dots , (x'_{k_t}, y_T) . The collection of edges of $G(\Theta, \Phi)$ constructed in this way is denoted by $\tilde{\Sigma}(\Theta)$. In an analogous way the paths in $\Sigma(\Phi)$ give rise to a set $\tilde{\Sigma}(\Phi)$ of edges of $G(\Theta, \Phi)$. These are to be the only edges of $G(\Theta, \Phi)$. Observe that $\tilde{\Sigma}(\Theta)$, resp. $\tilde{\Sigma}(\Phi)$, is a set of pairwise node disjoint edges of $G(\Theta, \Phi)$.

The rule of [3] can now be formulated:

If λ is a (finite) collection of pairwise node disjoint edges of $G(\Theta, \Phi)$ with

$$X' \subseteq \text{In } \lambda \subseteq X_I \cup X', \quad X'' \subseteq \text{Ter } \lambda \subseteq Y_T \cup X'',$$

then there is a (finite) collection Π of pairwise node disjoint finite linear paths in Γ with

$$\text{In } \Pi = \{x: x_I \in \text{In } \lambda\}, \quad \text{Ter } \Pi = \{y: y_T \in \text{Ter } \lambda\}.$$

As formulated in [3] the presence of cyclic paths (paths whose terminal node coincides with its initial node) is possible. The set Π above is what remains after the cyclic paths have been discarded.

3. Let X be an arbitrary set. A *matroid* [10], [11] on X is a non-empty collection \mathbf{M} of subsets of X such that (1) $A \in \mathbf{M}$, $A' \subseteq A$ imply $A' \in \mathbf{M}$ and (2) $A_1, A_2 \in \mathbf{M}$, $|A_1| + 1 = |A_2| < \infty$ imply there is an $x \in A_2 \setminus A_1$ with $A_1 \cup \{x\} \in \mathbf{M}$. Those subsets of X which are members of \mathbf{M} are called *independent sets*. A *base* of \mathbf{M} is a maximal in-

dependent subset of X . If $|X| < \infty$, \mathbf{M} is a *finite matroid*; if one base of \mathbf{M} is finite, then all bases are finite and have the same cardinal number. Matroids with a finite base are called *rank-finite matroids*. If $S \subseteq X$, then $\mathbf{M}_S = \{A \subseteq S: A \in \mathbf{M}\}$ is a matroid on S , called the restriction of \mathbf{M} to S . If \mathbf{M}_1 and \mathbf{M}_2 are matroids on disjoint sets X_1 and X_2 , respectively, then $\mathbf{M}_1 \oplus \mathbf{M}_2$ is a matroid on $X_1 \cup X_2$, called the *direct sum* of \mathbf{M}_1 and \mathbf{M}_2 . The matroid on a set X consisting of all subsets of X is called the *free matroid* on X and is denoted by $\mathcal{P}(X)$. For more information on these matters, one may consult [10], [11], or [5].

There are a number of ways to induce matroids from given matroids. Let G be a bipartite graph with node set $X \cup Y$ ($X \cap Y = \emptyset$) where each edge has initial node in X and terminal node in Y . Let \mathbf{M}^1 be a matroid on Y and define a collection \mathbf{M}^2 of subsets of X by: $A \in \mathbf{M}^2$ if and only if there is a set Π of pairwise node disjoint edges with $\text{In } \Pi = A$ and $\text{Ter } \Pi \in \mathbf{M}^1$. Then \mathbf{M}^2 is a matroid on X [7]. Those matroids obtained in this way when \mathbf{M}^1 is the free matroid on Y are called *transversal matroids* [4]. We use the rule formulated in §2 to give relatively simple proofs of the following theorems. But first we state and prove two lemmas.

If \mathbf{M} is a matroid on a set X and $A \in \mathbf{M}$, then the *span* of A is the set $A \cup \{z \in X \setminus A: A \cup \{z\} \in \mathbf{M}\}$. If Z is the span of A , then A is a base of \mathbf{M}_Z .

LEMMA 1. *Let G be a bipartite graph with node set $X \cup Y$ ($X \cap Y = \emptyset$), and let \mathbf{M}^1 be a matroid on Y . Let Θ_i be a finite set of pairwise node disjoint edges of G with $\text{In } \Theta_i = B_i$ and $\text{Ter } \Theta_i = A_i \in \mathbf{M}^1$ ($i = 1, 2$). Then one of the following two possibilities must occur.*

(1) *There exists an $x \in B_2 \setminus B_1$ and a set Π of pairwise node disjoint edges with $\text{In } \Pi = B_1 \cup \{x\}$ and $A_1 \cap A_2 \subseteq \text{Ter } \Pi \in \mathbf{M}^1$.*

(2) *There exists a set Π of pairwise node disjoint edges with $\text{In } \Pi = B_1$, $A_1 \cap A_2 \subseteq \text{Ter } \Pi \in \mathbf{M}^1$, and $A_2 \subseteq Z$ where Z is the span of $\text{Ter } \Pi$ (relative to the matroid \mathbf{M}^1).*

If A_2 is contained in the span of A_1 relative to \mathbf{M}^1 , then we may take $\Pi = \Theta_1$ to satisfy possibility (2). Thus assume there is a $y \in A_2 \setminus A_1$ with $A_1 \cup \{y\} \in \mathbf{M}^1$. The node y determines a sequence of distinct edges which alternate in Θ_2 and Θ_1 and which either (i) ends with an edge of Θ_2 with initial node $x \in B_2 \setminus B_1$ or (ii) ends with an edge of Θ_1 with terminal node $z \in A_1 \setminus A_2$. Let Δ_i be the set of edges of Θ_i in this sequence ($i = 1, 2$) and let $\Theta_3 = \{\Theta_1 \setminus \Delta_1\} \cup \Delta_2$. In case (i), Θ_3 is a set of pairwise node disjoint edges of G with $\text{In } \Theta_3 = B_1 \cup \{x\}$ and $A_1 \cap A_2 \subseteq \text{Ter } \Theta_3 = A_1 \cup \{y\} \in \mathbf{M}^1$. We may then take $\Pi = \Theta_3$ to satisfy possibility (1). In case (ii), Θ_3 is a set of pairwise node disjoint edges with

In $\Theta_3 = B_1$, $A_1 \cap A_2 \subseteq \text{Ter } \Theta_3 = \{A_1 \setminus \{y\}\} \cup \{z\} = A_3 \in \mathbf{M}^1$ where $|A_3 \cap A_2| > |A_1 \cap A_2|$. We may then replace Θ_1 by Θ_3 and repeat the above procedure. After a finite number of applications either we achieve case (i) and thus satisfy possibility (1) or we obtain a set Θ_k ($k \geq 3$) of pairwise node disjoint edges with $\text{In } \Theta_k = B_1$, $A_1 \cap A_2 \subseteq \text{Ter } \Theta_k = A_k \in \mathbf{M}^1$ where $A_k \subseteq A_2$. If A_2 is contained in the span of A_k relative to \mathbf{M}^1 , we may take $\Pi = \Theta_k$ to satisfy possibility (2). Otherwise one more application of the above procedure must lead to the occurrence of case (i) and thus to the satisfying of possibility (1).

LEMMA 2. *Let Γ be a directed graph, and let X, Y be subsets of the nodes of Γ . Assume that \mathbf{M}^1 is a matroid on Y . Let Θ_i be a finite set of pairwise node disjoint, finite linear paths in Γ with $\text{In } \Theta_i = B_i \subseteq X$ and $\text{Ter } \Theta_i = A_i \in \mathbf{M}^1$ ($i = 1, 2$). Then one of the following two possibilities must occur.*

- (1) *There exists an $x \in B_2 \setminus B_1$ and a set Π of pairwise node disjoint, finite linear paths in Γ with $\text{In } \Pi = B_1 \cup \{x\}$ and $\text{Ter } \Pi \in \mathbf{M}^1$.*
- (2) *There exists a set Π of pairwise node disjoint, finite linear paths in Γ with $\text{In } \Pi = B_1$, $\text{Ter } \Pi \in \mathbf{M}^1$, and $A_2 \subseteq Z$ where Z is the span of $\text{Ter } \Pi$ (relative to the matroid \mathbf{M}^1).*

We deduce Lemma 2 from Lemma 1. Thus let $G(\Theta_1, \Theta_2)$ be the bipartite graph associated with Θ_1 and Θ_2 as in §2. Because of the bijection from Y to Y_T we may consider \mathbf{M}^1 as a matroid on Y_T . We then consider the matroid $\mathbf{M}^1 \oplus \mathcal{P}(X'')$ on $Y_T \cup X''$. Then $\tilde{\Sigma}(\Theta_i)$ is a collection of pairwise node disjoint edges in $G(\Theta_1, \Theta_2)$ with $\text{In } \tilde{\Sigma}(\Theta_i) = (B_i)_I \cup X'$ and $\text{Ter } \tilde{\Sigma}(\Theta_i) = (A_i)_T \cup X'' \in \mathbf{M}^1 \oplus \mathcal{P}(X'')$ ($i = 1, 2$). We now apply Lemma 1 to conclude one of the following two possibilities must occur.

- (3) *There exists $x_I \in (B_2)_I \setminus (B_1)_I$ and a set λ of pairwise node disjoint edges of $G(\Theta_1, \Theta_2)$ with $\text{In } \lambda = (B_1)_I \cup \{x_I\} \cup X'$ and $X'' \subseteq \text{Ter } \lambda \in \mathbf{M}^1 \oplus \mathcal{P}(X'')$.*
- (4) *There exists a set λ of pairwise node disjoint edges of $G(\Theta_1, \Theta_2)$ with $\text{In } \lambda = (B_1)_I \cup X'$, $X'' \subseteq \text{Ter } \lambda \in \mathbf{M}^1 \oplus \mathcal{P}(X'')$, and $(A_2)_T \cup X'' \subseteq Z$ where Z is the span of $\text{Ter } \Pi$ (relative to the matroid $\mathbf{M}^1 \oplus \mathcal{P}(X'')$).*

By the rule of §2, (3) leads to an $x \in B_2 \setminus B_1$ and a set Π of pairwise node disjoint, finite linear paths satisfying (1), while (4) leads to a set Π satisfying (2).

We use Lemma 2 now to prove Theorems 1 and 2.

THEOREM 1. *Let Γ be a directed graph, and let X, Y be subsets of the nodes of Γ . Let \mathbf{M}^1 be a matroid on Y and define a collection \mathbf{M}^2 of subsets of X by: $B \in \mathbf{M}^2$ if and only if there is a set Θ of pairwise node disjoint,*

finite linear paths with $\text{In } \Theta = B$, $\text{Ter } \Theta \in \mathbf{M}^1$. Then \mathbf{M}^2 is a matroid on X .

Property (1) for matroids being clear, we need to verify property (2). Thus let $B_1, B_2 \in \mathbf{M}^2$ with $|B_1| = k$, $|B_2| = k+1$, and let Θ_i be a set of pairwise node disjoint, finite linear paths in Γ with $\text{In } \Theta_i = B_i$, $\text{Ter } \Theta_i = A_i \in \mathbf{M}^1$ ($i = 1, 2$). We now apply Lemma 2 and observe that possibility (2) can now not occur. For if it did we would have A_2 contained in the span of $\text{Ter } \Pi$ (relative to the matroid \mathbf{M}^1) where $A_2 \in \mathbf{M}$ and $|A_2| = k+1$ but $|\text{Ter } \Pi| = k$. This contradicts the defining properties of matroids. Hence possibility (1) of Lemma 2 must occur, and we conclude that there is an $x \in B_2 \setminus B_1$ with $B_1 \cup \{x\} \in \mathbf{M}^2$. Hence \mathbf{M}^2 is a matroid on X .

With reference to Theorem 1, the matroid \mathbf{M}^2 (on X) is said to be induced from the matroid \mathbf{M}^1 (on Y) by the directed graph Γ . If \mathbf{M}^1 is the free matroid on Y , this result has been proved in [8] and [9] (where a matroid is referred to as a pre-independence structure).

THEOREM 2. *Let Γ be a directed graph, and let X, Y be subsets of the nodes of Γ . Suppose the matroid \mathbf{M}^2 on X is induced from the rank-finite matroid \mathbf{M}^1 on Y by Γ . If B is any base of \mathbf{M}^2 and Θ is any set of pairwise node disjoint, finite linear paths in Γ with $\text{In } \Theta = B$, $\text{Ter } \Theta = A \in \mathbf{M}^1$, then \mathbf{M}^2 (on X) is induced from \mathbf{M}_Z^1 (on Z) by Γ where Z is the span of A relative to the matroid \mathbf{M}^1 .*

The corresponding result for bipartite graphs is derived in [2] when \mathbf{M}^1 is the free matroid on Y (Z is then A) and is generalized in [6]. If \mathbf{M}^3 is the matroid induced on X from \mathbf{M}_Z^1 on Z by Γ , then it is clear that $\mathbf{M}^3 \subseteq \mathbf{M}^2$. Thus we need to show that $\mathbf{M}^2 \subseteq \mathbf{M}^3$. This is a consequence of the following argument.

Let B_i be a base of \mathbf{M}^2 and let Θ_i be a set of pairwise node disjoint, finite linear paths in Γ with $\text{In } \Theta_i = B_i$, $\text{Ter } \Theta_i = A_i \in \mathbf{M}^1$ ($i = 1, 2$). We now apply Lemma 2 and observe that possibility (1) can now not occur. For, if it did, it would have as a consequence that there is an $x \in B_2 \setminus B_1$ with $B_1 \cup \{x\} \in \mathbf{M}^2$. This, however, contradicts the assumption that B_1 is a base of \mathbf{M}^2 . Hence possibility (2) of Lemma 2 must occur. Thus there is a set Π of pairwise node disjoint, finite linear paths in Γ with $\text{In } \Pi = B_1$, $\text{Ter } \Pi \in \mathbf{M}^1$, and $A_2 \subseteq Z$ where Z is the span of $\text{Ter } \Pi$ (relative to \mathbf{M}^1). But $A_2, \text{Ter } \Pi \in \mathbf{M}^1$ with $|A_2| = |B_2| = |B_1| = |\text{Ter } \Pi|$. But this means that $\text{Ter } \Pi$ is contained in the span of A_2 (relative to \mathbf{M}^1). Thus we have a set Π of pairwise node disjoint, finite linear paths in Γ with $\text{In } \Pi = B_1$ and $\text{Ter } \Pi$ contained in the span

of A_2 (relative to \mathbf{M}^2). Thus \mathbf{M}^2 (on X) is induced from $\mathbf{M}_{Z_2}^1$ (on Z_2) where Z_2 is the span of A_2 (relative to \mathbf{M}^2).

Let \mathbf{M} be a matroid on a set X . Then \mathbf{M} has been called *base orderable* if given any two bases B_1, B_2 there is a bijection $\sigma: B_1 \rightarrow B_2$ with $\{B_1 \setminus \{x\}\} \cup \{\sigma(x)\}$ and $\{B_2 \setminus \{\sigma(x)\}\} \cup \{x\}$ bases for all $x \in B_1$. Of course, the bijection σ must be the identity on $B_1 \cap B_2$. If in addition the bijection σ can be chosen so that $\{B_1 \setminus A\} \cup \sigma(A)$ and $\{B_2 \setminus \sigma(A)\} \cup A$ are bases for all $A \subseteq B_1$, then \mathbf{M} will be called *strongly base orderable*. (This definition may be unnecessary, for to our knowledge all known base orderable matroids are strongly base orderable. In what follows we require the seemingly stronger property.) It was proved in [2] that all transversal matroids are base orderable and indeed strongly base orderable. In [1] it was shown that not all matroids are base orderable, so that this property gives a true necessary condition for a matroid to be a transversal matroid. The result that transversal matroids are strongly base orderable was generalized in [6] to include all matroids induced from strongly base orderable matroids by a bipartite graph. Obviously a free matroid is strongly base orderable, and it is readily verified that a restriction of a strongly base orderable matroid is strongly base orderable. To show that a matroid induced from a strongly base orderable matroid by a directed graph is also strongly base orderable, we require the following lemma.

LEMMA 3. *Let G be a bipartite graph with node set $X \cup Y$ ($X \cap Y = \emptyset$), and let the matroid \mathbf{M}^2 on X be induced by G from the strongly base orderable, rank-finite matroid \mathbf{M}^1 on Y . If Θ_i is a set of pairwise node disjoint edges of G with $\text{In } \Theta_i = B_i$, a base of \mathbf{M}^2 , and $\text{Ter } \Theta_i = A_i \in \mathbf{M}^1$ ($i = 1, 2$), then there is a bijection $\rho: B_1 \setminus B_2 \rightarrow B_2 \setminus B_1$ such that for all $A \subseteq B_1 \setminus B_2$ there exists a set $\Theta_i(A)$ of pairwise node disjoint edges of G ($i = 1, 2$) with $\text{In } \Theta_1(A) = \{B_1 \setminus A\} \cup \rho(A)$, $A_1 \cap A_2 \subseteq \text{Ter } \Theta_1(A) \in \mathbf{M}^1$ and $\text{In } \Theta_2(A) = \{B_2 \setminus \rho(A)\} \cup A$, $A_1 \cap A_2 \subseteq \text{Ter } \Theta_2(A) \in \mathbf{M}^1$.*

Using Theorem 2 and the fact that the restriction of a strongly base orderable matroid is strongly base orderable, we conclude that we may assume that A_i is a base of \mathbf{M}^1 ($i = 1, 2$). Let $\sigma: A_1 \setminus A_2 \rightarrow A_2 \setminus A_1$ be a bijection with $\{A_1 \setminus W\} \cup \sigma(W)$ and $\{A_2 \setminus \sigma(W)\} \cup W$ bases of \mathbf{M}^1 for all $W \subseteq A_1 \setminus A_2$. We define a bijection $\rho: B_1 \setminus B_2 \rightarrow B_2 \setminus B_1$ as follows. Let $x \in B_1 \setminus B_2$. Then x determines a sequence of edges which alternate in Θ_1 and Θ_2 and which ends either with an edge of Θ_2 with initial vertex $y \in B_2 \setminus B_1$ or with an edge of Θ_1 with terminal vertex $w_1 \in A_1 \setminus A_2$. In the former case we define $\rho(x) = y$. In the latter case we consider $\sigma(w_1) \in B_2 \setminus A_2$ which determines a sequence of edges alternating in Θ_2 and Θ_1 that ends either with an edge of Θ_2 with initial node $y \in B_2 \setminus A_2$

or with an edge of Θ_1 with terminal node $w_2 \in A_1 \setminus A_2$ where $w_1 \neq w_2$. In the first case we define $\rho(x) = y$; in the second case we repeat with $\sigma(w_2)$ in place of $\sigma(w_1)$. After a finite number of steps we arrive at a finite sequence w_1, w_2, \dots, w_k ($k \geq 1$) of distinct nodes of $A_1 \setminus A_2$ such that the sequence of edges determined by $\sigma(w_k)$ which alternates in Θ_2 and Θ_1 ends with an edge of Θ_2 with initial node $y \in B_2 \setminus B_1$. We then define $\rho(x) = y$. Let $\Delta_i(x)$ be the set of edges of Θ_i encountered in defining $\rho(x)$ ($i = 1, 2$) and let $W(x) = \{w_1, w_2, \dots, w_k\}$. (Note that $W(x)$ may be empty.) Since ρ is a bijection, it is readily verified that $(\Delta_1(x) \cup \Delta_2(x) : x \in B_1 \setminus B_2)$ and $(W(x) : x \in B_1 \setminus B_2)$ are both families of pairwise disjoint sets. In particular, $\rho : B_1 \setminus B_2 \rightarrow B_2 \setminus B_1$ is an injection and thus a bijection. If $A \subseteq B_1 \setminus B_2$, then

$$\Theta_1(A) = \left\{ \Theta_1 \setminus \bigcup_{x \in A} \Delta_1(x) \right\} \cup \left\{ \bigcup_{x \in A} \Delta_2(x) \right\},$$

$$\Theta_2(A) = \left\{ \Theta_2 \setminus \bigcup_{x \in A} \Delta_2(x) \right\} \cup \left\{ \bigcup_{x \in A} \Delta_1(x) \right\}$$

are each pairwise node disjoint sets of edges of G with

$$\text{In } \Theta_1(A) = \{B_1 \setminus A\} \cup \rho(A),$$

$$A_1 \cap A_2 \subseteq \text{Ter } \Theta_1(A) = \left\{ A_1 \setminus \bigcup_{x \in A} W(x) \right\} \cup \sigma \left(\bigcup_{x \in A} W(x) \right) \in \mathbf{M}^1.$$

$$\text{In } \Theta_2(A) = \{B_2 \setminus \rho(A)\} \cup A,$$

$$A_1 \cap A_2 \subseteq \text{Ter } \Theta_2(A) = \left\{ A_2 \setminus \sigma \left(\bigcup_{x \in A} W(x) \right) \right\} \cup \left\{ \bigcup_{x \in A} W(x) \right\} \in \mathbf{M}^1.$$

This establishes the lemma.

THEOREM 3. *Let \mathbf{M}^1 be a strongly base orderable, rank-finite matroid on a set Y . If the matroid \mathbf{M}^2 on X can be induced from \mathbf{M}^1 by a directed graph Γ , then \mathbf{M}^2 is also strongly base orderable.*

Let Θ_i be a set of pairwise node disjoint, finite linear paths in Γ with $\text{In } \Theta_i = B_i$, a base of \mathbf{M}^2 , and $\text{Ter } \Theta_i = A_i \in \mathbf{M}^1$ ($i = 1, 2$). Let $G(\Theta_1, \Theta_2)$ be the bipartite graph associated with Θ_1 and Θ_2 as in §2, and regard \mathbf{M}^1 as a matroid on Y_T . Consider then the matroid $\mathbf{M}^1 \oplus \mathcal{P}(X'')$ on $Y_T \cup X''$. Then $\tilde{\Sigma}(\Theta_i)$ is a set of pairwise node disjoint edges of $G(\Theta_1, \Theta_2)$ with $\text{In } \tilde{\Sigma}(\Theta_i) = (B_i)_T \cup X'$, $\text{Ter } \tilde{\Sigma}(\Theta_i) = (A_i)_T \cup X''$ ($i = 1, 2$). Since from Theorem 2 we may assume that A_1 and A_2 are bases of \mathbf{M}^1 , it follows by a simple cardinality argument that $(B_i)_T \cup X'$ is a base of the matroid induced on $X_T \cup X'$ from $\mathbf{M}^1 \oplus \mathcal{P}(X'')$ by

$G(\Theta_1, \Theta_2)$. Using Lemma 3, we conclude that there is a bijection $\rho_I: (B_1)_I \setminus (B_2)_I \rightarrow (B_2)_I \setminus (B_1)_I$ such that for all $A_I \subseteq (B_1)_I \setminus (B_2)_I$ there are sets $\lambda_1(A)$, $\lambda_2(A)$ of pairwise node disjoint edges of $G(\Theta_1, \Theta_2)$ with

$$\begin{aligned} X' &\subseteq \text{In } \lambda_1(A) = \{(B_1)_I \setminus A_I\} \cup \rho_I(A_I), \\ X'' &\subseteq \text{Ter } \lambda_1(A) \in \mathbf{M}^1 \oplus \mathcal{P}(X''), \\ X' &\subseteq \text{In } \lambda_2(A) = \{(B_2)_I \setminus \rho_I(A_I)\} \cup A_I, \\ X'' &\subseteq \text{Ter } \lambda_2(A) \in \mathbf{M}^1 \oplus \mathcal{P}(X''). \end{aligned}$$

Thus by the rule of §2 there is a bijection $\rho: B_1 \setminus B_2 \rightarrow B_2 \setminus B_1$ with the property that for all $A \subseteq B_1 \setminus B_2$ there exist sets $\Pi_1(A)$, $\Pi_2(A)$ of pairwise node disjoint finite linear paths in Γ with

$$\begin{aligned} \text{In } \Pi_1(A) &= \{B_1 \setminus A\} \cup \rho(A), & \text{Ter } \Pi_1 &\in \mathbf{M}^1, \\ \text{In } \Pi_2(A) &= \{B_2 \setminus \rho(A)\} \cup A, & \text{Ter } \Pi_2 &\in \mathbf{M}^1. \end{aligned}$$

Hence \mathbf{M}^2 is a strongly base orderable matroid on X .

COROLLARY. *If a matroid \mathbf{M} on X can be induced from a free matroid by a directed graph, then \mathbf{M} is strongly base orderable.*

This is a special case of the theorem, since a free matroid is trivially strongly base orderable.

On six or fewer elements there are up to isomorphism only two matroids which are not transversal matroids. These are the cycle matroids [10] of the complete graph K_4 on four nodes, $\mathbf{M}(K_4)$, and the cycle matroid of the graph obtained from a triangle by repeating each edge. It is easy to verify that the latter matroid is strongly base orderable and indeed can be induced from a free matroid by a directed graph. The matroid $\mathbf{M}(K_4)$ is, however, not base orderable, as is easily verified, and hence cannot be induced from a free matroid by any directed graph. We may conclude that on six or fewer elements there is up to isomorphism exactly one matroid, $\mathbf{M}(K_4)$, which cannot be induced from a free matroid by a directed graph; similarly, there is exactly one matroid on six or fewer elements, namely $\mathbf{M}(K_4)$, which is not base orderable (or strongly base orderable).

REFERENCES

1. R. A. Brualdi, *Comments on bases in dependence structures*, Bull. Austral. Math. Soc. 1 (1969), 161–167. MR 40 #4146.
2. R. A. Brualdi and E. B. Scrimger, *Exchange systems, matchings, and transversals*, J. Combinatorial Theory 5 (1968), 244–257. MR 38 #55.
3. R. A. Brualdi and J. S. Pym, *A general linking theorem in directed graphs*,

edited by L. Mirsky, *Studies in Pure Mathematics*, Academic Press, London and New York, 1971.

4. J. Edmonds and D. R. Fulkerson, *Transversals and matroid partition*, J. Res. Nat. Bur. Standards Sect. B **69B** (1965), 147–153. MR **32** #5531.

5. F. Harary and D. J. A. Welsh, *Matroids versus graphs: The many facets of graph theory*, Lecture Notes in Math., no. 110, Springer-Verlag, Berlin, pp. 155–170.

6. J. H. Mason, *Representations of independence spaces*, Ph.D. Dissertation, University of Wisconsin, Madison, Wis., 1969.

7. H. Perfect, *Independence spaces and combinatorial problems*, Proc. London Math. Soc. (3) **19** (1969), 17–30. MR **39** #2649.

8. ———, *Applications of Menger's graph theorem*, J. Math. Anal. Appl. **22** (1968), 96–111. MR **37** #93.

9. J. S. Pym and H. Perfect, *Submodular functions and independence structures*, J. Math. Anal. Appl. **30** (1970), 1–37.

10. W. T. Tutte, *Lectures on matroids*, J. Res. Nat. Bur. Standards Sect. B **69B** (1965), 1–47. MR **31** #4023.

11. H. Whitney, *On the abstract properties of linear dependence*, Amer. J. Math. **57** (1935), 509–533.

UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706