ON RINGS SATISFYING $[(a, b, c), d] = 0$

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ABSTRACT. A simple nonassociative ring, in which the associators commute with all elements, is under mild additional assumptions either associative or commutative. This result cannot be extended to prime rings since a construction of semiprime rings gives counterexamples.

Introduction. Let R be a nonassociative ring, in which the associators commute with all elements. If R is simple then we prove under mild additional assumptions that R is either associative or commutative. This result cannot be extended to prime rings. To prove this we present a construction of semiprime rings giving counterexamples. Our class of rings includes the standard algebras and accessible rings, which were studied by Albert and Kleinfeld and others.

Structure theory. Let R be a nonassociative ring. We denote the deviation from the commutative law by the *commutator* $[x, y] := xy - yx$ and the deviation from the associative law by the *associator* $(x, y, z) := (xy)z - x(yz)$. In the following we study rings R satisfying the relation

$$(*) \quad [(a, b, c), d] = 0 \quad \text{for all } a, b, c, d \in R.$$

The relation $(*)$ holds in accessible and in standard rings [4]. If we put $V := \{x \mid x \in R, [x, y] = 0 \text{ for all } y \in R\}$, then $(*)$ is equivalent to saying that V contains all associators. In [4], [5], [6], [9], [10], [12], [13] other generalizations of standard rings are considered. In most rings that were studied until now, V is a subalgebra of R . Since we need this fact we state

LEMMA 1. *Let R be any ring. Then $V = \{x \mid x \in R, [x, R] = 0\}$ is a subring of V if R satisfies one of the following conditions:*

- (a) $(x, x, y) + (y, x, x) - 2(x, y, x) = 0$.
- (b) $(x, y, z) + (y, z, x) + (z, x, y) = 0, \quad x, y, z \in R.$
- (c) $(x, x, x) = 0$ and $2y = 0$ only for $y = 0$.

PROOF. The conditions (a), (b), (c) can be rewritten by using commutators respectively as:

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$$(a') \quad [x^2, y] + [[y, x], x] + [x, xy] + 3x[y, x] = 0.$$

$$(b') \quad [xy, z] + [yz, x] + [zx, y] = 0.$$

$$(c') \quad [x^2, x] = 0 \text{ and } 2y = 0 \text{ implies } y = 0.$$

From (b') or from the polarizations of (a') or (c') one easily sees that V is closed under multiplication, hence a subring of V . V was also used in the proofs of [3]. An easy example shows that V is not always a subalgebra of R . Clearly flexible rings, being defined by $(x, y, x) = 0$, satisfy (a). The rings of (γ, δ) -type [7] satisfy (b). In the following we will always assume that V is a subalgebra.

LEMMA 2. *If V is a subalgebra of R and R satisfies (*), then the set $W := \{v \mid v \in V, Rv \subset V\}$ is an ideal of R such that $(x, y, v) \in W$ and $(v, y, x) \in W$ for $v \in V$ and all $x, y \in R$.*

PROOF. From (*) we see that W is a (two-sided) ideal of R . From the well-known identity

$$a(b, c, d) + (a, b, c)d = (ab, c, d) - (a, bc, d) + (a, b, cd),$$

which holds in any ring, we get

$$z(x, y, v) = (zx, y, v) - (z, xy, v) + (z, x, yv) - (z, x, y)v \in V$$

since V is a subalgebra of R and (*) holds. Similarly we get $z(v, y, x) = (v, y, x)z \in V$.

COROLLARY 1. *The canonical homomorphism of R onto R/W maps V into the center of R/W .*

PROOF. Let $x, y \in R$ and $v \in V$. We know already $[x, v] = 0$, $(x, y, v) \in W$ and $(v, y, x) \in W$ from Lemma 2. Therefore also

$$\begin{aligned} (x, v, y) &= (x, y, v) + (v, x, y) - [xy, v] + [x, v]y - x[y, v] \\ &= (x, y, v) + (v, x, y) \in W. \end{aligned}$$

COROLLARY 2. *If $W = 0$, then V equals the center of R .*

PROOF. By Corollary 1, V is contained in the center while the other inclusion is trivial.

COROLLARY 3. $(x, y, z)^3 \equiv (x, x, x)(y, y, y)(z, z, z)$ and $2(x, y, z)^3 \equiv 0$ modulo W .

PROOF. Since V is mapped into the center of R/W , we have modulo W , that

$$\begin{aligned} (x, y, z)^3 &\equiv (x, y, z)(x(x, y, z), y, z) \equiv - (x, y, z)((x, x, y)z, y, z) \\ &\equiv - (x(x, x, y), y, z)(z, y, z) \equiv (x, x, x)(y, y, z)(z, y, z). \end{aligned}$$

Now

$$\begin{aligned}(y, y, z)(z, y, z) &\equiv (z, y(y, y, z), z) \equiv - (z, (y, y, y)z, z) \\ &\equiv - (y, y, y)(z, z, z)\end{aligned}$$

or

$$\begin{aligned}(y, y, z)(z, y, z) &\equiv (z(y, y, z), y, z) \equiv - ((z, y, y)z, y, z) \\ &\equiv - (z, (z, y, y)y, z) \equiv (z, z(y, y, y), z) \equiv (y, y, y)(z, z, z)\end{aligned}$$

which shows $2(y, y, y)(z, z, z) \equiv 0$.

THEOREM 1. *Let R be a ring without nonzero ideals $\neq R$ satisfying (*). If V is a subalgebra of R and (x, x, x) is nilpotent for each $x \in R$, then R is either associative or commutative.*

PROOF. The ideal W of Lemma 2 is contained in the commutative subalgebra V of R . Since R has no nontrivial ideal either $W = R$ or $W = 0$. In the first case R is commutative. Let us therefore consider the case $W = 0$. By Corollary 2 all associators are in the center of R and are nilpotent by Corollary 3. Therefore $R(x, y, z)$ is a nilpotent ideal of R . Hence $R(x, y, z) = 0$. This implies $(x, y, z) \in W$ or $(x, y, z) = 0$.

REMARK. In Lemma 1 we saw that the assumption that V is a subalgebra of R is a very mild restriction. From the proof of Corollary 3 we see $2(x, x, x)^2 = 0$. Hence the assumption that (x, x, x) is nilpotent is only necessary in case $2y = 0$ for all $y \in R$.

COROLLARY. *Let R be a simple ring satisfying (*). Then R is commutative or associative if one of the following conditions is satisfied:*

- (a'') $(x, x, y) + (y, x, x) - 2(x, y, x) = 0$ and $2z = 0$ implies $z = 0$.
- (b'') $(x, y, z) + (y, z, x) + (z, x, y) = 0$.
- (c'') $(x, x, x) = 0$ and $2z = 0$ implies $z = 0$.
- (d'') $(x, y, x) = 0$.

Construction of semiprime rings. In the preceding section we classified the simple rings satisfying (*). In similar cases, e.g. for rings with commutators in the nucleus, one can make a similar statement with simple weakened to prime or even semiprime [5], [6], [9], [12], [13]. This is not true for rings satisfying (*). We will give a construction of prime and semiprime rings to get counterexamples.

Let us recall that a ring is called *prime* if the product of any two (two-sided) nonzero ideals is nonzero. We call a ring *semiprime*, if the square of any nonzero ideal is nonzero. Clearly prime rings are semiprime.

Let k be a field and R a k -algebra with unit element e . We assume that R has a subvectorspace $M \neq 0$ containing all commutators such that $R = ke + M$ is a direct sum. Let ku be the k -vectorspace generated by some element $u \in R$. On the direct sum $R^u := ku + R$ we define a k -algebra structure by conserving the multiplication in R and requiring $u^2 = u$, $eu = ue = u$, $um = mu = 0$ for $m \in M$. It is clear that R^u is then a k -algebra with unit element e having R as a subalgebra and ku as an ideal such that $(ku)^2 = ku$. Some of the properties of R^u we state in

THEOREM 2. (i) *If R is simple, then R^u is prime.*

(ii) *If R is semiprime, then R^u is semiprime.*

(iii) *If $\sum_{i=1}^n [f_i(x_1, \dots, x_m), g_i(x_{m+1}, \dots, x_p)] h_i(x_{p+1}, \dots, x_r) = 0$ is a relation holding on R , then it also holds on R^u .*

(iv) *If $f(x_1, \dots, x_n)$ is a relation on R , then $[f(x_1, \dots, x_n), x_{n+1}] = 0$ is a relation on R^u .*

PROOF. Since ku is an ideal the only nontrivial statements are (i) and (ii). Let $I \neq 0$ be an ideal of R^u and $x = \alpha e + \beta u + \delta m \in I$, $x \neq 0$. Then $y := x - ux = \alpha(e - u) + \delta m \in I$. In case $R \cap I = 0$ we have $ny = yn = 0$ for each $n \in M$. From $y^2 = \alpha y$ we see $I^2 \neq 0$ for $\alpha \neq 0$. In case $\alpha = 0$ we have $y = \delta m \in R \cap I = 0$ or $x = \beta u$ and $u \in I$. Thus $R \cap I = 0$, $I \neq 0$, implies $I^2 \neq 0$. If R is semiprime, then $R \cap I \neq 0$ also implies $I^2 \neq 0$ since $R \cap I$ is a nonzero ideal of R . This proves (ii). If R is simple, then $R \cap I \neq 0$ implies $I = R^u$. In case $R \cap I = 0$, $I \neq 0$, we see from $ny = yn = 0$ for $n \in M$ that $k(\alpha e + \delta m)$ is an ideal of R . Hence either $R = ke$, $m = 0$ or $\alpha e + \delta m = 0$. In the first case $M = 0$ which was excluded. Hence $\beta u \in I$, $\beta \neq 0$, and we see $I = ku$. Since the only ideals of R^u are 0 , ku , R^u we conclude that R^u is prime.

COROLLARY. *If R satisfies one of the following relations, then R^u satisfies the same relation.*

(a) $(x, x, y) + (y, x, x) - 2(x, y, x) = 0$.

(b) $(x, y, z) + (y, z, x) + (z, x, y) = 0$.

(c) $(x, x, x) = 0$.

(d) $(x, y, x) = 0$ (flexibility).

(e) $[x, y] = 0$ (commutativity).

PROOF. We have only to remark that as in Lemma 1 each of the relations can be rewritten by using commutators so that (iii) applies.

For simple R we have $MM \not\subseteq M$ and we can find $m, m_i, n_i \in M$ such that $e = m + \sum m_i n_i$. This shows $\sum (m_i, n_i, u) = u$, $\sum (u, m_i, n_i) = -u$ and hence R^u is not associative and not alternative for $\text{char } k \neq 2$. But by (iv) of Theorem 2 for associative R the relation (*) holds on R^u .

Now let $R \neq ke$ be a simple finite-dimensional associative algebra,

e.g. a quaternion algebra, over k . Then R has a nonzero subspace M containing all commutators such that $R = ke + M$ is direct. The corresponding algebra R^* is prime, flexible, and satisfies (*) but is neither commutative nor associative. This shows that Theorem 1 cannot be extended to prime rings.

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