THE NUMBER OF BINOMIAL COEFFICIENTS DIVISIBLE BY A FIXED POWER OF 2

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ABSTRACT. Define $f(n, j)$ as the number of binomial coefficients $\binom{n}{r}$ divisible by exactly 2^j . We find formulas for $f(n, j)$ for $0 \leq j \leq 4$ and evaluate $f(n, j)$ for special values of n .

1. Introduction. In [1] L. Carlitz considered the problem of determining the number of binomial coefficients $\binom{n}{r}$ ($r=0, 1, \dots, n$) divisible by a fixed power of a prime p . In this paper we consider the same problem for $p=2$. We note that this problem is also discussed briefly in [3].

Let n be a positive integer and define $f(n, j)$ as the number of binomial coefficients $\binom{n}{r}$ ($r=0, 1, \dots, n$) divisible by exactly 2^j . We shall find formulas for $f(n, j)$ for $j=0, \dots, 4$, and we shall evaluate $f(n, j)$ for special values of n . Several of the results of [1] for $p=2$ are generalized in this paper.

Throughout this paper we shall assume that

$$(1.1) \quad n = 2^{e_1} + 2^{e_2} + \dots + 2^{e_m}, \quad 0 \leq e_1 < e_2 < \dots < e_m.$$

We shall also use the following notation:

$$e_1 - e_0 = k \text{ means } e_1 = k - 1,$$

$$e_1 - e_0 > k \text{ means } e_1 > k - 1.$$

We also define any binomial coefficient $\binom{x}{y}$ to be 0 if x or y is negative or if $x < y$.

2. Evaluation of $f(n, j)$. If n satisfies (1.1) and

$$(2.1) \quad \begin{aligned} r &= 2^{b_1} + \dots + 2^{b_v}, & 0 \leq b_1 < \dots < b_v, \\ n - r &= 2^{c_1} + \dots + 2^{c_w}, & 0 \leq c_1 < \dots < c_w, \end{aligned}$$

then it is well known that the exponent of the highest power of 2 dividing $\binom{n}{r}$ is $v+w-m$ [2, p. 263]. It is clear, then, that for $j > 0$,

$$(2.2) \quad f(n, j) = a_{j1}2^{m+j-2} + a_{j2}2^{m+j-4} + \dots + a_{jj}2^{m+j-2j}$$

where a_{ji} is the number of integers r such that (2.1) holds, $v+w=m+j$, and there are a total of $m+j-2i$ b_h 's and c_k 's such that

$$b_h \neq c_s \quad (s = 1, \dots, w), \quad c_k \neq b_s \quad (s = 1, \dots, v).$$

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In other words a_{ji} is the number of possible ways of doing the following: Select i elements from $e_1, \dots, e_m$. Call them $d_1, \dots, d_i$ and for $k=1, \dots, i$ write

$$2^{d_k} = 2^{d_k - g(k)} + \sum_{s=1}^{g(k)} 2^{(d_k - s)}$$

where $g(k)$ is a positive integer $\leq d_k$ and

$$\sum_{k=1}^i g(k) = j.$$

Let $\{x_1, \dots, x_{m-i}\} = \{e_1, \dots, e_m\} - \{d_1, \dots, d_i\}$ and divide the $m+j$ numbers

$$x_1, \dots, x_{m-i}, d_1 - 1, \dots, d_1 - g(1), \\ d_1 - g(1), \dots, d_i - 1, \dots, d_i - g(i), d_i - g(i)$$

into two sets $\{b_1, \dots, b_v\}$ and $\{c_1, \dots, c_w\}$ such that $0 \leq b_1 < \dots < b_v$, $0 \leq c_1 < \dots < c_w$ and $v+w=m+j$.

We can use this idea to evaluate $f(n, r)$. Suppose (1.1) holds and

$$\begin{aligned} e_i - e_{i-1} &> s \quad \text{for } q_s \text{ terms } e_i \quad (s = 1, \dots, 4); \\ e_i - e_{i-1} &= 1 \quad \text{and } e_{i-1} - e_{i-2} > 1 \quad \text{for } q_5 \text{ terms } e_i, \\ &= 2 \quad \text{for } q_6 \text{ terms } e_i, \\ &> 2 \quad \text{for } q_7 \text{ terms } e_i, \\ &> 3 \quad \text{for } q_8 \text{ terms } e_i; \\ e_i - e_{i-1} &= 2 \quad \text{and } e_{i-1} - e_{i-2} > 1 \quad \text{for } q_9 \text{ terms } e_i, \\ &> 2 \quad \text{for } q_{10} \text{ terms } e_i; \\ e_i - e_{i-1} &= 1, e_{i-1} - e_{i-2} = 1 \quad \text{and } e_{i-2} - e_{i-3} > 1 \quad \text{for } q_{11} \text{ terms } e_i, \\ &> 2 \quad \text{for } q_{12} \text{ terms } e_i; \\ e_i - e_{i-1} &= 1, e_{i-1} - e_{i-2} = 2, e_{i-2} - e_{i-3} > 1 \quad \text{for } q_{13} \text{ terms } e_i; \\ e_i - e_{i-1} &= 2, e_{i-1} - e_{i-2} = 1, e_{i-2} - e_{i-3} > 1 \quad \text{for } q_{14} \text{ terms } e_i; \\ e_i - e_{i-1} &= 1, e_{i-1} - e_{i-2} = 1, e_{i-2} - e_{i-3} = 1, \\ &e_{i-3} - e_{i-4} > 1 \quad \text{for } q_{15} \text{ terms } e_i; \\ e_i - e_{i-1} &= 3, e_{i-1} - e_{i-2} > 1 \quad \text{for } q_{16} \text{ terms } e_i. \end{aligned}$$

Then we have

$$(2.3) \quad f(n, 0) = 2^m,$$

$$(2.4) \quad f(n, 1) = q_1 2^{m-1},$$

$$(2.5) \quad f(n, 2) = q_2 2^m + \left(q_5 + \binom{q_1}{2} \right) 2^{m-2},$$

$$(2.6) \quad \begin{aligned} f(n, 3) = & q_3 2^{m+1} + (q_2(q_1 - 1) + q_9 + q_7) 2^{m-1} \\ & + \left(q_{11} + q_5(q_1 - 1) + \binom{q_1}{3} \right) 2^{m-3}, \end{aligned}$$

$$(2.7) \quad \begin{aligned} f(n, 4) = & q_4 2^{m+2} + \left(q_3(q_1 - 1) + \binom{q_2}{2} + q_8 + q_{10} + q_{16} \right) 2^m \\ & + \left(q_2 \binom{q_1 - 1}{2} + q_{12} + q_{14} + q_7(q_1 - 1) \right. \\ & \left. + q_9(q_1 - 2) + q_{13} + q_6 q_2 + q_7(q_2 - 1) \right) 2^{m-2} \\ & + \left(\binom{q_1}{4} + q_{15} + \binom{q_5}{2} + q_5 \binom{q_1 - 1}{2} \right. \\ & \left. + q_{11}(q_1 - 1) \right) 2^{m-4}. \end{aligned}$$

For example, if $n = 2^4 + 2^5 + 2^{20} + 2^{26} + 2^{28}$ we have $q_1 = 4$, $q_2 = 3$, $q_3 = 3$, $q_4 = 3$, $q_5 = 1$, $q_7 = 1$, $q_8 = 1$, $q_9 = 1$, $q_{10} = 1$, all other q 's = 0, and therefore

$$\begin{aligned} f(n, 0) &= 2^5, \quad f(n, 1) = 4 \cdot 2^4, \\ f(n, 2) &= 3 \cdot 2^5 + 7 \cdot 2^3, \quad f(n, 3) = 3 \cdot 2^6 + 11 \cdot 2^4 + 7 \cdot 2^2, \\ f(n, 4) &= 3 \cdot 2^7 + 14 \cdot 2^5 + 16 \cdot 2^3 + 4 \cdot 2. \end{aligned}$$

We note that the formulas for $f(n, 0)$ and $f(n, 1)$ can be found in [1] and [3].

3. Some special evaluations. Carlitz points out in [1] that if $e_i - e_{i-1} = 1$ for $i = 1, \dots, m$ then $f(n, j) = 0$ for $j > 0$. Some of the other results of [1] are special cases of the following theorems, which we prove by using the method of §2.

THEOREM 3.1. *If (1.1) holds, $e_i - e_{i-1} = 1$ for $m-1$ terms e_i and $e_{r+1} - e_r = t+1$, $t > 0$, for some $r \geq 0$, and $m-r \leq t$, then*

$$\begin{aligned} f(n, 0) &= 2^m, \\ f(n, j) &= 2^{m-j} + 2^{m-j+2} + \dots + 2^{m-j+2(j-1)} \\ &\quad (j = 1, \dots, m-r), \end{aligned}$$

$$\begin{aligned}
 f(n, j) &= 2^{m+j-2} + 2^{m+j-4} + \dots + 2^{m+j-2(m-r)} \\
 &\quad (j = m - r + 1, \dots, t), \\
 f(n, t + m - r - j) &= 2^{r+t+j-2} + 2^{r+t+j-4} + \dots + 2^{r+t+j-2j} \\
 &\quad (j = 1, \dots, m - r - 1), \\
 f(n, j) &= 0 \quad \text{if } j > t + m - r - 1.
 \end{aligned}$$

If $m - r > t$, then

$$\begin{aligned}
 f(n, j) &= 2^{m-j} + 2^{m-j+2} + \dots + 2^{m-j+2(j-1)} \\
 &\quad (j = 1, \dots, t), \\
 f(n, t + j) &= 2^{m+t-(j+2)} + 2^{m+t-(j+4)} + \dots + 2^{m+t-(j+2t)} \\
 &\quad (j = 1, \dots, m - r - t - 1), \\
 f(n, m - r + j) &= 2^{r+j} + 2^{r+j+2} + \dots + 2^{r+j+2(t-j-1)} \\
 &\quad (j = 0, \dots, t - 1), \\
 f(n, j) &= 0 \quad \text{if } j > t + m - r - 1.
 \end{aligned}$$

For example, if $n = 2^t$, then

$$f(n, 0) = 2, \quad f(n, j) = 2^{j-1} \quad (0 < j \leq t).$$

THEOREM 3.2. If (1.1) holds, and $e_1 = u > 0$, $e_i - e_{i-1} = 1$ for $i = 2, \dots, m-1$, $e_m - e_{m-1} = 2$, $m-1 \geq u$, then

$$\begin{aligned}
 f(n, 0) &= 2^m, \\
 f(n, 1) &= 2^m, \\
 f(n, j) &= 2^{m+j-2} + 2 \cdot 2^{m+j-4} + \dots + 2 \cdot 2^{m+j-2j} \\
 &\quad (j = 2, \dots, u), \\
 f(n, u + j) &= 2^{m+u-j-1} + 2^{m+u-j-3} + \dots + 2^{m+u-j-(2u-1)} \\
 &\quad (j = 1, \dots, m - u - 1), \\
 f(n, m) &= 1 + 2 \cdot 2^2 + \dots + 2 \cdot 2^{2(u-1)}, \\
 f(n, m + j) &= 2^{j+1} + 2^{j+3} + \dots + 2^{j+2(u-j)-1} \\
 &\quad (j = 1, \dots, u - 2), \\
 f(n, u + m - 1) &= 2^u \quad (\text{if } u > 1), \\
 f(n, u + m) &= 2^u.
 \end{aligned}$$

If $m-1 < u$, then

$$\begin{aligned}
 f(n, j) &= 2^{m+j-2} + 2 \cdot 2^{m+j-4} + \dots + 2 \cdot 2^{m+j-2j} \\
 &\quad (j = 2, \dots, m - 1),
 \end{aligned}$$

$$\begin{aligned}
 f(n, m) &= 1 + 2 \cdot 2^2 + \dots + 2 \cdot 2^{2(m-2)} + 2^{2m-2}, \\
 f(n, m+j) &= 2^{2m+j-2} + 2 \cdot 2^{2m+j-4} + \dots + 2 \cdot 2^{2m+j-2m} \\
 &\quad (j = 1, \dots, u-m), \\
 f(n, u+j) &= 2^{m+u-j-1} + 2^{m+u-j-3} + \dots + 2^{u+m-j-(2m-2j-1)} \\
 &\quad (j = 1, \dots, m-2), \\
 f(n, u+m-1) &= 2^u \quad (\text{if } u > 1), \\
 f(n, u+m) &= 2^u.
 \end{aligned}$$

In all of the following theorems, we assume that (1.1) holds and $f(n, j)$ is given by (2.2). Theorems 3.3 and 3.4 are proved in [3].

THEOREM 3.3. *If $e_i - e_{i-1} > j$ for q terms e_i and $e_i - e_{i-1} = j$ for $m-q$ terms e_i , then*

$$\begin{aligned}
 a_{j1} &= q, \\
 a_{ji} &= \binom{m}{i} \binom{j-1}{i-1} \quad (i = 2, \dots, j).
 \end{aligned}$$

THEOREM 3.4. *If $e_i - e_{i-1} > j$ for q terms e_i , $e_i - e_{i-1} = j$ for $m-q-1$ terms e_i , and $e_1 = u$, $0 \leq u < j-1$, then*

$$\begin{aligned}
 a_{j1} &= q, \quad a_{ji} = \binom{m}{i} \binom{j-1}{i-1} - \binom{m-1}{i-1} \binom{j-u-1}{i-1} \\
 &\quad (i = 2, \dots, j).
 \end{aligned}$$

To get more results of this type, we use the method described in §2. We also use the fact that the number of compositions of j into i parts, $j = x_1 + x_2 + \dots + x_i$, such that $x_1 > u_1$, $x_2 > u_2$, $\dots$, $x_k > u_k$ is

$$\binom{j - u_1 - \dots - u_k - 1}{i-1}.$$

See [4, p. 124] and [3].

THEOREM 3.5. *If $e_i - e_{i-1} > j$ for q terms e_i , $e_i - e_{i-1} = j$ for $m-q-1$ terms e_i and $e_k - e_{k-1} = u < j$ for some $k \neq 1$, then*

$$\begin{aligned}
 a_{j1} &= q, \\
 a_{ji} &= \binom{m}{i} \binom{j-1}{i-1} - \binom{j-u-1}{i-1} \binom{m-2}{i-2} - \binom{m-2}{i-1} \binom{j-u}{i-1} \\
 &\quad (i = 2, \dots, j).
 \end{aligned}$$

THEOREM 3.6. *If $e_i - e_{i-1} > j$ for q terms e_i , $e_i - e_{i-1} = j$ for $m - q - 2$ terms e_i and $e_1 = v < j - 1$, $e_k - e_{k+1} = u < j$ for some $k > 2$, then*

$$a_{j1} = q,$$

$$\begin{aligned} a_{ji} = & \binom{m}{i} \binom{j-1}{i-1} - \binom{j-u-1}{i-1} \binom{m-2}{i-2} \\ & - \binom{m-2}{i-1} \binom{j-u}{i-1} - \binom{j-v-1}{i-1} \binom{m-1}{i-1} \\ & + \binom{j-u-v}{i-1} \binom{m-3}{i-2} + \binom{j-u-v-1}{i-1} \binom{m-3}{i-3} \\ & (i = 2, \dots, j). \end{aligned}$$

THEOREM 3.7. *If $e_i - e_{i-1} > j$ for q terms e_i , $e_i - e_{i-1} = j$ for $m - q - 2$ terms e_i , $e_2 - e_1 = u < j$, $e_1 = v < j - 1$, then*

$$a_{j1} = q,$$

$$\begin{aligned} a_{ji} = & \binom{m}{i} \binom{j-1}{i-1} - \binom{m-2}{i-1} \binom{j-u}{i-1} - \binom{m-2}{i-2} \binom{j-u-1}{i-1} \\ & - \binom{m-1}{i-1} \binom{j-v-1}{i-1} + \binom{m-2}{i-2} \binom{j-u-v-1}{i-1} \\ & (i = 2, \dots, j). \end{aligned}$$

THEOREM 3.8. *If $e_i - e_{i-1} > j$ for q terms e_i , $e_i - e_{i-1} = j$ for $m - q - 2$ terms e_i , $e_{k+1} - e_k = u < j$, $e_k - e_{k-1} = v < j$ for some $k > 1$, then*

$$a_{j1} = q,$$

$$\begin{aligned} a_{ji} = & \binom{m}{i} \binom{j-1}{i-1} - \binom{m-2}{i-2} \left(\binom{j-u-1}{i-1} + \binom{j-v-1}{i-1} \right) \\ & - \binom{m-2}{i-1} \left(\binom{j-u}{i-1} + \binom{j-v}{i-1} \right) \\ & + \binom{j-u-v-1}{i-1} \binom{m-3}{i-3} + \binom{j-u-v}{i-1} \binom{m-3}{i-2} \\ & (i = 2, \dots, j). \end{aligned}$$

THEOREM 3.9. *If $e_i - e_{i-1} > j$ for q terms e_i , $e_i - e_{i-1} = j$ for $m - q - 2$ terms e_i , $e_k - e_{k-1} = u < j$ for some $k > 1$, $e_t - e_{t-1} = v < j$ for some $t > k + 1$, then*

$$a_{j1} = q,$$

$$\begin{aligned} a_{ji} = & \binom{m}{i} \binom{j-1}{i-1} - \binom{j-u-1}{i-1} \binom{m-2}{i-2} - \binom{j-u}{i-1} \binom{m-2}{i-1} \\ & - \binom{j-v-1}{i-1} \binom{m-2}{i-2} - \binom{j-v}{i-1} \binom{m-2}{i-1} \\ & + \binom{j-u-v+1}{i-1} \binom{m-4}{i-2} + 2 \binom{j-u-v}{i-1} \binom{m-4}{i-3} \\ & + \binom{j-u-v-1}{i-1} \binom{m-4}{i-4} \quad (i = 2, \dots, j). \end{aligned}$$

For example, again consider $n = 2^4 + 2^5 + 2^{20} + 2^{26} + 2^{28}$ and suppose we wish to find $f(n, 4)$. Using Theorem 3.9, we have $j = 4$, $q = 3$, $u = 1$, $v = 2$, $m = 5$, and so

$$f(n, 4) = 3 \cdot 2^7 + 14 \cdot 2^5 + 16 \cdot 2^3 + 4 \cdot 2.$$

Finally, we determine the positive integer s such that $f(n, s) \neq 0$ and $f(n, j) = 0$ for $j > s$. Suppose

$$e_i - e_{i-1} \leq k \quad (i = 1, \dots, m)$$

and suppose

$$e_i - e_{i-1} = 1 \quad (i = 1, \dots, r), \quad e_{r+1} - e_r > 1.$$

We note that $r = 0$ if $e_1 > 0$. If for $i > r$, $e_i - e_{i-1} = t$ for h_i terms e_i ($t = 1, \dots, k$), then it is clear by the remarks of §2 that

$$f(n, j) = 0 \quad \text{if } j > h_1 + 2h_2 + \dots + kh_k - 1 = e_m - r$$

and $f(n, e^m - r) = 2^{h_2 + 2h_3 + \dots + (k-1)h_k + r - 1}$.

For example, if $n = 2^4 + 2^5 + 2^{20} + 2^{26} + 2^{28}$ then $r = 0$, $h_1 = h_2 = h_5 = h_6 = h_{15} = 1$ and $f(n, j) = 0$ if $j > 28$, $f(n, 28) = 2^{23}$.

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