AN EXTENSION OF THE THOMSON PRINCIPLE

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ABSTRACT. The classical Thomson principle, giving lower bounds for the Dirichlet integral, is extended, with modification, to a wider class of test functions. This makes it possible to obtain simpler and better lower bounds.

1. Introduction. Consider a Jordan region A in m -dimensional Euclidean space E_m , with exterior normal n on the boundary A^* . Let y be a real-valued function and ψ an m -vector-valued function on the closure of A , each continuous and piecewise continuously differentiable.

The Thomson principle (see [2, p. 43] for a particular version) states that, if $\operatorname{div} \psi = 0$ (*), then

$$\int_A y'^2 d\mu \geq \left(\int_{A^*} y\psi \cdot n \, ds \right)^2 / \int_A \psi^2 d\mu$$

(the second integral is over the boundary hypersurface). This result gives lower bounds for the Dirichlet integral $\int_A y'^2 d\mu$, which can be used, for example, in estimation of capacitance.

In this paper, we shall relax the condition (*) on ψ . This allows us to obtain very simple lower bounds, and to improve particular lower bounds obtained under condition (*).

This work is a by-product of similar work on lower bounds in control systems on multidimensional domains [3], carried out under a Faculty Research Fellowship at Western Michigan University.

2. Modified Thomson principle.

THEOREM. If $I^* \operatorname{div} \psi \leq 0$ and $y|_{A^*} \geq 0$, then

$$\int_A y'^2 d\mu \geq I^{*2}/I$$

where $I = \int_A \psi^2 d\mu$, $I^* = \int_{A^*} y\psi \cdot n \, ds$.

PROOF. By the Dirichlet principle,

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$$\int_A y'^2 d\mu \geq \int_A Y'^2 d\mu$$

where $Y|_{A^*} = y|_{A^*}$ and Y is harmonic with minimum on A^* . By the divergence theorem,

$$\int_{A^*} Y\psi \cdot n \, ds = \int_A (Y' \cdot \psi + Y \operatorname{div} \psi) \, d\mu.$$

Hence

$$\begin{aligned} \int_A Y'^2 d\mu &= \int_A (Y'^2 + Y' \cdot \psi + Y \operatorname{div} \psi) \, d\mu - \int_{A^*} Y\psi \cdot n \, ds \\ &\geq L(\psi) = -\frac{1}{4} \int_A \psi^2 d\mu - \int_{A^*} y\psi \cdot n \, ds \end{aligned}$$

provided $\operatorname{div} \psi \geq 0$. Now

$$\begin{aligned} L(t\psi) &= -\frac{1}{4}t^2I - tI^* = -I(\tfrac{1}{2}t + I^*/I)^2 + I^{*2}/I \\ &= I^{*2}/I \quad \text{when } t = -2I^*/I, \end{aligned}$$

so $\int_A y'^2 d\mu \geq I^{*2}/I$ provided $\frac{1}{2}I \operatorname{div} t\psi = -I^* \operatorname{div} \psi \geq 0$.

REMARK 1. If $\operatorname{div} \psi = 0$, the condition $y|_{A^*} \geq 0$ is unnecessary, and we get the classical Thomson principle.

REMARK 2. We can extend our result to a combination of linearly independent functions ψ_i , with possible improvement of the lower bound. In $L(\psi) = -\frac{1}{4}\int_A \psi^2 d\mu - \int_{A^*} y\psi \cdot n \, ds$, put $\psi = t_i \psi_i$ (with repeated subscripts indicating summation). With

$$\begin{aligned} I_{ij} &= \int_A \psi_i \cdot \psi_j d\mu, \quad J = [I_{ij}]^{-1}, \quad \text{and} \quad I_i^* = \int_{A^*} y\psi_i \cdot n \, ds, \\ L(\psi) &= -\frac{1}{4}t_i I_{ij} t_j - t_i I_i^* \\ &= -(\tfrac{1}{2}t_i + I_h^* J_{hi}) I_{ij} (\tfrac{1}{2}t_j + J_{jk} I_k^*) + I_j^* J_{jk} I_k^* \\ &= I_j^* J_{jk} I_k^* \quad \text{when } t_j = -2J_{jk} I_k^*. \end{aligned}$$

Note that the matrix $[I_{ij}]$ is nonsingular since $t_i I_{ij} t_j = \int_A (t_i \psi_i)^2 d\mu > 0$ if $[t_i] \neq 0$. The condition on ψ now reads $\operatorname{div} \psi_j J_{jk} I_k^* \leq 0$.

3. Estimation of capacitance. Let A be the region exterior to a closed hypersurface A_1^* and interior to a second closed hypersurface A_0^* . We can calculate the capacitance of such regions by minimizing the Dirichlet integral $\int_A y'^2 d\mu$ over those y for which $y|_{A_1^*} = 1$, $y|_{A_0^*} = 0$.

To obtain a lower bound, take ψ_i with components $(\psi_i)_k = 0$ for $k \neq i$, x_i for $k = i$. The ψ_i are obviously independent. Then $I_{ij} = 0$ for

$i \neq j$, $\int_A x_i^2 d\mu$ for $i=j$; and $I_i^* = -\mu_1$ where μ_1 is the measure of the region A_1 enclosed by A_1^* . Here

$$\operatorname{div} \psi_j J_{jk} I_k^* = -\mu_1 \sum 1 / \int_A x_i^2 d\mu < 0.$$

Thus we have the crude but simple lower bound

$$\mu_1^2 \sum 1 / \int_A x_i^2 d\mu.$$

Coordinate axes should be chosen to make this as large as possible, that is, along the principal axes of the positive matrix $[\int_A x_i x_j d\mu]$, with origin at the centroid of A .

As another choice for ψ , consider $\psi = r^k x$ where $r = |x|$, $0 \in A_1$. Then $I = \int_A r^{2k+2} d\mu$;

$$\begin{aligned} I^* &= -(m+k) \int_{A_1} r^k d\mu \quad \text{for } k > -m, \\ &= -s_m \quad \text{for } k = -m, \end{aligned}$$

where s_m is the $(m-1)$ -dimensional measure of the unit sphere in E_m ; and $\operatorname{div} \psi = (m+k)r^k$, so $\operatorname{div} \psi I^* \leq 0$ for $k \geq -m$. Thus, for $k > -m$, we have the lower bound

$$L_k = (m+k)^2 \left(\int_{A_1} r^k d\mu \right)^2 / \int_A r^{2k+2} d\mu.$$

4. An example. Let A in E_3 be the ellipsoidal shell

$$\{x: a \leq r(c^2 \sin^2 \phi + 1)^{1/2} \leq b\}$$

where ϕ is the latitudinal coordinate. Consider $\int_A y'^2 d\mu$ for $y=1$ on the inner boundary and $y=0$ on the outer boundary. Here

$$\begin{aligned} \int_{A_1} r^k d\mu &= 4\pi K_{(k+3)/2} a^{k+3} / (k+3), \\ \int_A r^{2k+2} d\mu &= 4\pi K_{(2k+5)/2} (b^{2k+5} - a^{2k+5}) / (2k+5) \end{aligned}$$

where $K_p = \int_0^1 (c^2 x^2 + 1)^{-p} dx$, for which we have the recurrence relation

$$K_p = K_{p-1}(2p-3)/(2p-2) + (1+c^2)^{1-p}/(2p-2)$$

with $K_0 = 1$, $K_{1/2} = (\sinh^{-1} c)/c$.

By §3, we have the lower bounds

$$L_k = 4\pi K_{(k+3)/2}^2 (2k+5) a^{2k+6} / K_{(2k+5)/2} (b^{2k+5} - a^{2k+5})$$

for $k > -3$. For the case $k = -3$ in which $\operatorname{div} \psi = 0$,

$$L_{-3} = 8\pi ab/(b-a) [(1+c^2)^{1/2} + (\sinh^{-1} c)/c]$$

so $cL_{-3} \rightarrow 8\pi ab/(b-a)$ as $c \rightarrow \infty$. For the case $k = -2$ in which $\operatorname{div} \psi \neq 0$,

$$L_{-2} = 4\pi a^2 (\sinh^{-1} c)/(b-a)c$$

so $cL_{-2} \rightarrow \infty$ as $c \rightarrow \infty$. Thus for large c , $k = -2$ gives a better lower bound than the classical $k = -3$.

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