AN ASYMPTOTIC SOLUTION OF A NONHOMOGENEOUS LINEAR SYSTEM OF DIFFERENTIAL EQUATIONS¹

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ABSTRACT. A complete asymptotic series expansion is found for the solutions of a nonhomogeneous linear system of differential equations whose coefficient matrix and forcing term are in $L^1[t_0, \infty)$. Related results, which require that the coefficients satisfy certain conditional integrability conditions, are shown to be reducible by a transformation to the L^1 case.

In this note we will obtain an asymptotic series expansion for the solutions of the nonhomogeneous linear system of differential equations

$$(1) \quad dx/dt = A(t)x + h(t)$$

whenever A and h satisfy certain integrability conditions. In (1), A is a continuous $n \times n$ matrix that is defined on the interval $J = [t_0, \infty)$ and h is an n -vector that is continuous on J . The symbol $\|\cdot\|$ will denote some convenient norm of a vector and its corresponding matrix norm.

THEOREM 1. *Suppose that the conditions*

$$(2) \quad \int_0^\infty \|A(\tau)\| d\tau < \infty$$

and

$$(3) \quad \int_0^\infty \|h(\tau)\| d\tau < \infty$$

are satisfied. Then, any solution $x = x(t)$ of (1) can be written as

$$x(t) = X(t)d + p(t), \quad t \in J,$$

for some constant vector d , where

$$(4) \quad X(t) = \sum_{i=0}^{\infty} (-1)^i R^i(t)$$

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with

$$R^0(t) = I_n \quad (I_n \text{ is the } n \times n \text{ identity matrix}),$$

$$R^i(t) = \int_t^\infty A(\tau) R^{i-1}(\tau) d\tau, \quad i = 1, 2, \dots,$$

and

$$(5) \quad p(t) = \sum_{i=0}^{\infty} (-1)^i p^i(t)$$

with

$$p^0(t) = - \int_t^\infty h(\tau) d\tau, \quad p^i(t) = \int_t^\infty A(\tau) p^{i-1}(\tau) d\tau, \quad i = 1, 2, \dots$$

PROOF. The series representation for the fundamental matrix, $X(t)$, of solutions of the homogeneous system associated with equation (1) has been previously obtained by M. Ráb [5]. By virtue of the principle of superposition, we only need to show that $x(t) = p(t)$ is a solution of (1).

First, we observe that the series (5) converges uniformly on J . Let $\epsilon(t)$ be defined by the equation

$$\epsilon(t) = \int_t^\infty \|A(\tau)\| d\tau$$

and

$$h_0 = \int_{t_0}^\infty \|h(\tau)\| d\tau;$$

then, induction establishes the inequality

$$\|p^i(t)\| \leq h_0 \frac{\epsilon^i(t)}{i!}, \quad i = 0, 1, 2, \dots$$

The series (5) is therefore majorized by the series for $h_0 \exp \epsilon(t)$. Since $\epsilon(t)$ is bounded on J , it follows that (5) is uniformly convergent on J .

A solution of the integral equation

$$(6) \quad x(t) = - \int_t^\infty A(\tau) x(\tau) d\tau - \int_t^\infty h(\tau) d\tau$$

is also a solution of (1). The proof of the theorem will be complete if it is shown that (5) is a solution of (6). An application of the Moore-Osgood iterated limits theorem yields the equation

$$\begin{aligned}
p(t) &= \sum_{i=0}^{\infty} (-1)^i p^i(t) = - \int_t^{\infty} h(\tau) d\tau - \sum_{i=1}^{\infty} (-1)^{i-1} \int_t^{\infty} A(\tau) p^{i-1}(\tau) d\tau \\
&= - \int_t^{\infty} h(\tau) d\tau - \int_t^{\infty} A(\tau) \sum_{i=1}^{\infty} (-1)^{i-1} p^{i-1}(\tau) d\tau \\
&= - \int_t^{\infty} h(\tau) d\tau - \int_t^{\infty} A(\tau) p(\tau) d\tau;
\end{aligned}$$

this completes the proof of the theorem.

If (2) or (3) is not satisfied, it is still possible to find asymptotic series solutions of (1) in certain instances when the integral $\int_t^{\infty} A(\tau) d\tau$ converges conditionally. Let M be any matrix such that $\int_t^{\infty} M(\tau) d\tau < \infty$, then the function M^{∞} will be defined as

$$M^{\infty}(t) = \int_t^{\infty} M(\tau) d\tau, \quad t \in J.$$

We will also use the following notation in the corollaries below. If $\int_t^{\infty} h(\tau) d\tau < \infty$, let $k_0(t) = -\int_t^{\infty} h(\tau) d\tau$; if $\int_t^{\infty} h(\tau) d\tau = \infty$, let $k_0(t) = \int_{t_0}^t h(\tau) d\tau$.

Suppose that there exists a positive integer c such that the integrals $k_i(\infty)$, $i = 0, 1, 2, \dots, c-1$, diverge, where

$$k_0(t) = \int_{t_0}^t h(\tau) d\tau, \quad k_i(t) = \int_{t_0}^t A(\tau) k_{i-1}(\tau) d\tau, \quad i = 1, 2, \dots, c-1;$$

but the integral $\int_t^{\infty} A(\tau) k_{c-1}(\tau) d\tau$ converges (possibly conditionally). We define

$$k_{c+i}(t) = - \int_t^{\infty} A(\tau) k_{c+i-1}(\tau) d\tau, \quad i = 0, 1, 2, \dots, a-c,$$

where a is an integer not less than c such that

$$(7) \quad \int_t^{\infty} \|A(\tau) k_{a-1}(\tau)\| d\tau < \infty.$$

A simple example of a pair of functions A and h having the above structure with $c=1$ and $a=2$ is $A(t) = e^{-t} \sin t^2$ and $h(t) = e^t$.

In the next corollary, we will use a condition, due to A. Wintner [6], which implies that the homogeneous equation associated with (1) has linear asymptotic equilibrium. It should be remarked that Wintner only alludes to the general integrability condition used here and does not indicate the complete transformation that is required except for the case $M=0$ below. (See [6, p. 187, line 8].)

We will use the following notation: let $C_0(t) = A(t)$, $D_0(t) = I_n + C_0^\infty(t)$ and define inductively,

$$C_m(t) = C_{m-1}^\infty(t)C_{m-1}(t)D_{m-1}^{-1}(t), \quad D_m(t) = I_n + C_m^\infty(t),$$

$$m = 1, 2, \dots$$

It will be assumed that the above integrals are well defined and that $D_m(t)$ is invertible for all t in J .

COROLLARY 1. *Let (7) hold and suppose that*

$$(8) \quad \int^\infty \|C_M^\infty(\tau)C_M(\tau)\| d\tau < \infty,$$

for some positive integer M . Then, any solution $x = x(t)$ of (1) can be written as

$$(9) \quad x(t) = E_M^{-1}(t) \left\{ \sum_{i=0}^\infty (-1)^i [R_M^i(t)d + \dot{p}_M^i(t)] \right\} + \sum_{j=0}^{a-1} k_j(t)$$

for some constant d , where $E_M(t) = D_M(t)D_{M-1}(t) \cdots D_1(t)D_0(t)$, the sequence $\{R_M^i\}$ is given by (4) with A replaced by C_{M+1} , and the sequence $\dot{p}_M^i$ is given by (5) with $h = E_MA k_{a-1}$.

PROOF. The transformation $x = E_M^{-1}u + \sum_{j=0}^{a-1} k_j$ in (1) leads to the equation

$$(10) \quad du/dt = [E_M(t)A(t) + dE_M(t)/dt]E_M^{-1}(t)u + E_M(t)A(t)k_{a-1}(t).$$

From the definition of E_M and C_i , it follows that

$$\begin{aligned} -dE_M/dt &= C_M D_{M-1} D_{M-2} \cdots D_1 D_0 + D_M C_{M-1} D_{M-2} \cdots D_1 D_0 \\ &\quad + D_M D_{M-1} C_{M-2} D_{M-3} \cdots D_1 D_0 \\ &\quad + \cdots + D_M D_{M-1} \cdots D_2 C_1 D_0 + D_M D_{M-1} \cdots D_1 C_0 \\ &= C_M^\infty C_{M-1}^\infty \cdots C_0^\infty C_0 + C_{M-1}^\infty C_{M-2}^\infty \cdots C_0^\infty C_0 \\ &\quad + D_M C_{M-2}^\infty C_{M-3}^\infty \cdots C_0^\infty C_0 + D_M D_{M-1} C_{M-3}^\infty \cdots C_0^\infty C_0 \\ &\quad + \cdots + D_M D_{M-1} \cdots D_2 C_0^\infty C_0 + D_M D_{M-1} \cdots D_1 C_0 \\ &\quad - C_M^\infty C_{M-1}^\infty \cdots C_0^\infty C_0 \\ &= D_M D_{M-1} \cdots D_1 D_0 C_0 - C_M^\infty C_{M-1}^\infty \cdots C_0^\infty C_0. \end{aligned}$$

Using this result in (10), we obtain

$$du/dt = C_{M+1}(t)u + E_M(t)A(t)k_{a-1}(t).$$

As a consequence of conditions (7) and (8), the coefficients in the above equation are in $L^1[J]$. Theorem 1 can be applied to find the

series expansion for u ; the transformation then leads to (9).

The first term of the transformation used in Corollary 1 was suggested by the work of Wintner [6]. The second term was suggested by some work of T. Manacorda [4]; both Corollary 1 and Corollary 2 below extend the result in [4] in several ways.

Now, define the sequence $\{F_i\}$ as

$$(11) \quad F_0 = A, \quad F_i = F_{i-1}^\infty A, \quad i = 1, 2, \dots$$

Denote by $G_m = I_n + \sum_{i=0}^m F_i^\infty$; we will suppose that t_0 is sufficiently large so that G_m is invertible on J .

COROLLARY 2. *Let (7) hold and suppose that*

$$(12) \quad \int^\infty \|F_{N+1}(\tau)\| d\tau < \infty$$

for some nonnegative integer N . Then, any solution $x = x(t)$ of (1) may be written:

$$(13) \quad x(t) = \sum_{i=0}^\infty (-1)^i G_N^{-1}(t) [R_N^i(t)d + p_N^i(t)] + \sum_{j=0}^{a-1} k_j(t)$$

for some constant d , where the sequence R_N^i is given by (4) with A replaced by $F_{N+1}G_N^{-1}$ and the sequence p_N^i is given by (5) with $h = G_N A k_{a-1}$.

PROOF. Applying the transformation $x = G_N^{-1}u + \sum_{j=0}^{a-1} k_j$ to equation (1) leads to the differential equation

$$(14) \quad du/dt = F_{N+1}(t)G_N^{-1}(t)u + G_N(t)A(t)k_{a-1}(t).$$

Since F_{N+1} is in $L^1[J]$ then so is $F_{N+1}G_N^{-1}$; also, by (7), $G_N A k_{a-1}$ is in $L^1[J]$. Theorem 1 may now be applied to (14); this leads to the expansion (13).

Corollary 2 improves the result in [3] in several ways. Namely, we obtain an asymptotic series expansion for the solutions of a non-homogeneous system as opposed to a single term asymptotic representation for the solutions of a homogeneous system. Also, the proof given here, when specialized to the case investigated in [3], is more compact since a simple transformation reduces the problem to a known result.

The referee has kindly remarked that if the integral $\int^\infty \|A(\tau)\| d\tau$ converges and the integral $\int^\infty \|h(\tau)\| d\tau$ does not exist but condition (7) is satisfied, then a simpler conclusion can be obtained than results by using Corollary 1 (with $C_1 = A$) or Corollary 2 (with $F_1 = A$). In this setting, the transformation $x = u + \sum_{j=0}^{a-1} k_j$ can be used to show that any solution x of (1) possesses the asymptotic series expansion

$$x(t) = \sum_{i=0}^{\infty} (-1)^i [R^i(t)d + p^i(t)] + \sum_{j=0}^{a-1} k_j(t)$$

where p^i is given by (5) with $h = Ak_{a-1}$.

The above corollaries may be used in conjunction with another transformation to obtain new asymptotic forms for the solutions of equation (1) when it has been subjected to a linear perturbation. Such problems, which consider the differential equation

$$(15) \quad dy/dt = A(t)y + P(t)y + h(t),$$

have been discussed in [1], [2], [5]. The substitution $y = X(t)u$ takes (15) into the equation

$$du/dt = Q(t)u + X^{-1}(t)h(t)$$

where $Q = X^{-1}PX$. If the coefficient function Q and the forcing function $X^{-1}h$ satisfy the hypotheses of either of the corollaries then new results are obtained for this perturbation problem.

We also point out, as is done in [3] and [6], that by the use of an adjoint system, asymptotic series expansions for the solutions of (1) may be derived by using a commuted form of the hypotheses of the corollaries. For example, in Corollary 1, the commuted analogue of hypothesis (8),

$$\int_0^{\infty} \|C_M(\tau)C_M^{\infty}(\tau)\| d\tau < \infty,$$

may be used to obtain an asymptotic series expansion for the solutions of (1).

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