A NOTE CONCERNING $A^* = L_1(\mu)$

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ABSTRACT. It is shown that there are exactly two abstract L -spaces which are duals of infinite dimensional separable Banach spaces.

A problem of long standing is to characterize those Banach spaces of the type $L_1(\mu)$ which are isomorphic to a dual Banach space. One of the early negative results was Gelfand's theorem that the space $L_1[0, 1]$ is not isomorphic to a dual space. Recently Pelczynski has given an extension of this result to the case $L_1(\mu)$ for μ a σ -finite non-purely atomic measure [6]. In a recent paper [8], Rosenthal has given some deep results on the structure of $C(X)^*$ for certain compact Hausdorff spaces X and a generalization of Pelczynski's result.

The main purpose of this note is to show that there are exactly two possible spaces of the type $L_1(\mu)$ which can be the dual of an infinite dimensional separable Banach space. This is closely related to some results in [8, p. 242].

NOTATION. Let m be a cardinal number and S a set of cardinal m . The L -space $l_1(m)$ is the Banach space of all absolutely summable real-valued functions on S with $\|f\| = \sum_{s \in S} |f(s)|$ for all $f \in l_1(m)$. The L -space $L_1[0, 1]$ is the classical space of all Lebesgue integrable functions on $[0, 1]$ with the norm $\|f\| = \int_0^1 |f(t)| dt$ for all $f \in L_1[0, 1]$. If $\{X_i\}_{i \in I}$ is a family of Banach spaces, $[\oplus \sum_I X_i]_1$ denotes the Banach space whose elements are the indexed sets $\{x_i\}_{i \in I}$ with $x_i \in X_i$ for all $i \in I$ and $\sum_{i \in I} \|x_i\| < \infty$. This last number is the norm of $\{x_i\}_{i \in I}$. All Banach spaces in this note are over the real number field and infinite dimensional. By an L -space it is meant a space of the type $L_1(\mu)$.

The following lemma gives a linear norm preserving extension of A^* into B^* when $A \subset B$ and A^* is an L -space.

LEMMA 1. *Let A and B be Banach spaces with $A \subset B$ and A^* an L -space. Then there is a linear isometry $T: A^* \rightarrow B^*$ such that $T(x^*)(x) = x^*(x)$ for all $x^* \in A^*$ and $x \in A$.*

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PROOF. Let $T: A^* \rightarrow A^{***}$, $K: B \rightarrow B^{**}$, $k: A \rightarrow B$ be the natural maps. Since A^* is an L -space, A^{**} is a P_1 space. Hence there is a contractive projection P on B^{**} with range $k^{**}(A^{**})$. Let $T = [(k^{**})^{-1}PK]^* J$. Then T has the required properties.

Before stating the theorem, the essential property of the compact Hausdorff spaces under consideration is discussed. Let X be a compact Hausdorff space such that for each regular Borel measure μ on X , $L_1(\mu)$ is separable. Examples of such spaces are the compact metric spaces and the weakly compact subsets of Banach spaces (see [3]). This last class includes spaces of arbitrarily high cardinal. Also, the dispersed spaces have this property since every regular Borel measure on a dispersed space is purely atomic.

THEOREM. *Let X be a compact Hausdorff space with the above property and A a closed linear subspace of X . If $A^* = L_1(\mu)$ where μ is not purely atomic, then A^* is linearly isometric to $(l_1(m) \oplus \sum_n L_1[0, 1])_1$ for some cardinals m and n with $m, n \geq c$, the cardinality of the continuum.*

PROOF. By a standard argument involving Zorn's lemma and the decomposition of L -spaces into direct sums of subspaces with weak order units, it can be assumed that

$$A^* = \left[l_1(m) + \left(\oplus_{n \in N} L_1(\mu_n) \right)_1 \right]_1$$

where m is a cardinal and N is a set of cardinals and μ_n is a finite nonatomic measure for each $n \in N$. If $L_1(\mu_n)$ is not separable, then it contains a copy of some nonseparable Hilbert space H [9], and by the lemma $C(X)^*$ contains a copy of H . This is impossible by Lemma 1.3 of [8]. Thus $L_1(\mu_n)$ is separable and since μ_n is nonatomic, $L_1(\mu_n) = L_1[0, 1]$ for each $n \in N$.

Since A^* is not of the type $l_1(a)$, it follows from [1] that there is a separable subspace B of A such that $B^* = L_1(\nu)$ and ν is not purely atomic. Hence, by [2], B contains a subspace C such that $C^* = C[0, 1]^*$. Since every extreme point of the unit sphere of C^* is the image of an extreme point in the unit sphere of A^* and m = cardinality of the extreme points of the unit sphere of A^* , $m \geq c$.

Now, it is well known that

$$C[0, 1]^* = \left[l_1(c) + \left(\oplus_c L_1[0, 1] \right)_1 \right]_1$$

(in fact, see [8] for a more general result). Since each $L_1[0, 1]$ contains

a copy of a separable Hilbert space H , by the lemma there is a linear isometry T of $(\oplus \sum_{s \in S} H_s)_1$ into A^* where S is a set of cardinality c and H_s is the same isomorphic copy of H for all $s \in S$. Let P be the natural projection in A^* with range $l_1(m)$ and null space $(\oplus \sum_{n \in N} L_1(\mu_n))_1$. Then $PT|H_s$ is compact for each $s \in S$ from the appendix in [9]. Thus there is an infinite-dimensional subspace H'_s of H_s such that $\|PT|H'_s\| < \frac{1}{2}$. Hence $\|(I-P)Th\| \geq \frac{1}{2}\|h\|$ for all h in $(\oplus \sum_{s \in S} H'_s)_1$. That is, $(\oplus \sum_{n \in N} L_1(\mu_n))_1$ contains an isomorphic copy of $(\oplus \sum_{s \in S} H'_s)_1$. Since the smallest cardinal of a dense set in $(\oplus \sum_{s \in S} H'_s)$ is c , it follows that $\text{card } N \geq c$.

SUMMARY. If A is a separable Banach such that $A^* = L_1(\mu)$, then

(a) $A^* = l_1 = l_1(\aleph_0)$ if and only if the set of extreme points in the unit sphere of A is countable;

(b) $A^* = [l_1(c) + (\oplus \sum_c L_1[0, 1])_1]$ if and only if the set of extreme points of the unit sphere of A^* is uncountable.

The necessity of (a) is clear. The sufficiency follows from the fact that the unit sphere of A^* is weak* metrizable and, hence, every element of the unit sphere is "represented" by a measure supported on the extreme points (see [7]). It is noted that an incorrect proof of this result appears in [5].

The necessity of (b) is clear and the sufficiency follows from the theorem since $A \subset C[0, 1]$.

Using a different argument, it can actually be shown that if

$$A^* = \left[l_1(m) + \left(\oplus \sum_{n \in N} L_1(\mu_n) \right)_{1 \rightarrow 1} \right]$$

with $N \neq \emptyset$, then $m \geq c$ and $\text{card } N \geq c$.

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