A CHARACTERIZATION OF REGULAR LOCAL RINGS

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ABSTRACT. For a local ring, $(A, \mathfrak{M})$ of positive depth regularity is shown to be equivalent to the symmetric algebra of $\mathfrak{M}$ being torsion free.

Let A be a commutative ring with unit element and $\mathfrak{A}$ an ideal of A . The symmetric algebra $S(\mathfrak{A})$ has been an object of study in [1], [2], and [4], particularly with regard to the questions of when it is factorial or simply a domain. In the special case where A is a (Noetherian) local ring and $\mathfrak{A} = \mathfrak{M}$, its maximal ideal, Micali [1] has shown that $S(\mathfrak{M})$ is a domain if and only if A is regular. It is the purpose of this note to prove the following somewhat different characterization of regular local rings.

THEOREM. *Let A be a local ring with maximal ideal $\mathfrak{M}$, depth $A > 0$. Then A is regular if and only if $S(\mathfrak{M})$ is a torsion free A -module.*

NOTE. An A -module M is called torsion free if $\ker(M \rightarrow M \otimes_A B) = 0$ where B is the total ring of fractions of A .

Before proceeding with the proof, we set down some notation in the general case. Suppose that $\mathfrak{A} = (a_1, \dots, a_n)$. Then the A -epimorphism $A^n \rightarrow \mathfrak{A}$ given by $(b_1, \dots, b_n) \mapsto \sum_{i=1}^n b_i a_i$ induces an A -algebra epimorphism $\mu: A[X_1, \dots, X_n] \rightarrow S(\mathfrak{A})$. If we denote the kernel of μ by $\mathfrak{Q}$, then $\mathfrak{Q}$ is the homogeneous ideal generated by all linear forms $\sum_{i=1}^n b_i X_i$ such that $\sum_{i=1}^n b_i a_i = 0$.

One also has the notion of the Rees algebra, $R(\mathfrak{A})$, of the ideal $\mathfrak{A}$. This is defined as $R(\mathfrak{A}) = A[a_1 X, \dots, a_n X] \subset A[X]$. The map $\mathfrak{A} \rightarrow R(\mathfrak{A})$ given by $a \mapsto aX$ induces an A -algebra epimorphism $\phi: S(\mathfrak{A}) \rightarrow R(\mathfrak{A})$. Finally one has the A -algebra epimorphism $\gamma: A[X_1, \dots, X_n] \rightarrow R(\mathfrak{A})$ given by $X_i \mapsto a_i X$. The kernel of γ is denoted by $\mathfrak{Q}_\infty$. It is the homogeneous ideal generated by all forms $f(X_1, \dots, X_n)$ such that $f(a_1, \dots, a_n) = 0$. It is easily seen that $\gamma = \phi \circ \mu$ and so $\ker \phi \approx \mathfrak{Q}_\infty / \mathfrak{Q}$.

These maps can be embedded in a slightly larger commutative diagram which is a useful framework for viewing things. Thus we have

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$$\begin{array}{ccc}
 & & R(\mathfrak{A}) \\
 & \nearrow \gamma & \uparrow \phi \\
 A[X_1, \dots, X_n] & \xrightarrow{\mu} & S(\mathfrak{A}) \\
 \downarrow \alpha & & \downarrow \eta \\
 (A/\mathfrak{A})[X_1, \dots, X_n] & \xrightarrow{\lambda} & S_{A/\mathfrak{A}}(\mathfrak{A}/\mathfrak{A}^2)
 \end{array}$$

where α is just reduction of coefficients mod $\mathfrak{A}$ and λ and η are the obvious maps coming from the universal properties of symmetric algebras. We denote the kernel of α by $\mathfrak{A}^*$.

REMARK. Suppose that $a_1, \dots, a_n$ is a minimal generating set for the maximal ideal $\mathfrak{M}$ in a local ring A . Then:

(1) λ is an isomorphism. For if k is the residue field $A/\mathfrak{M}$, then $[\mathfrak{M}/\mathfrak{M}^2:k] = n$.

(2) $\mathfrak{Q}_\infty \subset \mathfrak{M}^*$ if and only if $a_1, \dots, a_n$ are analytically independent [3, p. 67].

The theorem will now follow from the following lemma.

LEMMA. *Let $\mathfrak{A}$ be a finitely generated ideal of A (not necessarily a domain) with grade $\mathfrak{A} > 0$. Then $\ker \phi$ is contained in the torsion submodule of $S(\mathfrak{A})$.*

PROOF. Suppose $\mathfrak{A} = (a_1, \dots, a_n)$ where we can assume that a_n is a regular element. Denote by $\mathfrak{Q}_\infty^{(k)}$ the forms of degree k in $\mathfrak{Q}_\infty$. Since $\ker \phi \approx \mathfrak{Q}_\infty/\mathfrak{Q}$, it suffices to show that

$$a_n^{\binom{k}{2}} \mathfrak{Q}_\infty^{(k)} \subset \mathfrak{Q} \quad \text{for all } k \geq 1$$

where we interpret

$$a_n^{\binom{1}{2}} = 1.$$

This is done by induction on k and since $\mathfrak{Q}_\infty^{(1)} \subset \mathfrak{Q}$, the induction is begun. Assume then that

$$a_n^{\binom{k-1}{2}} \mathfrak{Q}_\infty^{(k-1)} \subset \mathfrak{Q}$$

and let $f \in \mathfrak{Q}_\infty^{(k)}$. One can write

$$f = X_1 f_1(X_1, \dots, X_n) + X_2 f_2(X_2, \dots, X_n) + \dots + X_n f_n(X_n)$$

where $f_1, \dots, f_n$ are forms of degree $k-1$. Set

$$g = X_1 f_1(a_1, \dots, a_n) + X_2 f_2(a_2, \dots, a_n) + \dots + X_n f_n(a_n).$$

Then $g \in \mathfrak{Q}_\infty^{(1)} \subset \mathfrak{Q}$. Consider now the form

$$h = a_n^{k-1} f - X_n^{k-1} g = X_1 h_1(X_1, \dots, X_n) + \dots + X_{n-1} h_{n-1}(X_{n-1}, X_n)$$

where

$$h_i(X_i, \dots, X_n) = a_n^{k-1} f_i(X_i, \dots, X_n) - X_n^{k-1} f_i(a_i, \dots, a_n).$$

Thus every $h_i \in \mathfrak{Q}_\infty^{(k-1)}$. By the induction hypothesis

$$a_n^{\binom{k-1}{2}} h \in \mathfrak{Q}$$

and so

$$a_n^{k-1+\binom{k-1}{2}} f = a_n^{\binom{k}{2}} f \in \mathfrak{Q},$$

completing the induction.

PROOF OF THEOREM. The fact that $S(\mathfrak{M})$ is torsion free, if A is regular follows immediately from Micali's theorem.

Suppose now that $S(\mathfrak{M})$ is torsion free. Since $\text{depth } A > 0$, i.e., $\text{grade } \mathfrak{M} > 0$, we have by the Lemma, $\mathfrak{Q}_\infty \subset \mathfrak{Q}$. On the other hand, $\mathfrak{Q} \subset \ker(\lambda \circ \alpha) = \mathfrak{M}^*$ by the above remark. Therefore a minimal generating set of $\mathfrak{M}$ is analytically independent. By a well-known result [3, p. 72, Theorem 4], A is regular.

REMARK. Since $\text{depth } A = 0$ implies that every A -module is torsion free, the existence of nonregular local rings of depth 0 shows that the theorem cannot be extended to this case.

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