A CLASS OF COMPARISON THEOREMS FOR NON-SELFADJOINT ELLIPTIC EQUATIONS

KURT KREITH¹

ABSTRACT. Seemingly disparate comparison theorems for nonselfadjoint elliptic equations have been established by C. A. Swanson and the author. The present paper establishes a class of comparison theorems involving an arbitrary vector valued function $\mathbf{p}(x)$. Special choices of $\mathbf{p}(x)$ yield both of the comparison theorems cited above.

Several papers have recently concerned themselves with generalizations of Sturm's comparison theorem to nonselfadjoint elliptic equations. The first such theorem was due to Protter [1], and subsequent generalizations have been established by Swanson [2] and the author [3]. An interesting aspect of [2] and [3] is the seemingly disparate hypotheses which the principal theorems require.

The purpose of this paper is to establish a class of comparison theorems for nonselfadjoint elliptic equations which contain the results of [2] and [3] as special cases. Our method consists of combining appropriate generalizations of techniques used by Picard [4] and Picone [5].

Let D denote a smooth bounded domain in R^n and let $u(x)$ and $v(x)$ be nontrivial real valued solutions of the elliptic equations

$$(1) \quad - \sum_{i,j=1}^n \frac{\partial}{\partial x_j} \left(a_{ij}(x) \frac{\partial u}{\partial x_i} \right) + 2 \sum_{i=1}^n b_i(x) \frac{\partial u}{\partial x_i} + c(x)u = 0,$$

$$(2) \quad - \sum_{i,j=1}^n \frac{\partial}{\partial x_j} \left(\alpha_{ij}(x) \frac{\partial v}{\partial x_i} \right) + 2 \sum_{i=1}^n \beta_i(x) \frac{\partial v}{\partial x_i} + \gamma(x)v = 0.$$

It is assumed that the coefficients $a_{ij}(x)$, $\alpha_{ij}(x)$, $b_i(x)$ and $\beta_i(x)$ are of class C' in $\bar{D}$ while $c(x)$ and $\gamma(x)$ are to be continuous in $\bar{D}$ (some of these requirements can be weakened somewhat as will be clear from the manipulations below). Furthermore (1) and (2) are to be uniformly elliptic in $\bar{D}$.

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Given a solution $u(x)$ of (1), we shall describe its behavior on ∂D by means of the equation $\partial u / \partial n + s(x)u = 0$ on ∂D , where

$$\frac{\partial u}{\partial n} = \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial n}{\partial x_j}$$

denotes the transverse derivative of u and $\partial n / \partial x_j$ denotes the cosine of the angle between the exterior unit normal n and the positive x_j -axis. The symbol $s(x_0) = \infty$ will be used to denote the boundary condition $u(x_0) = 0$. The boundary behavior of a solution $v(x)$ of (2) will similarly be described by $\partial v / \partial \nu + \sigma(x)v = 0$, where

$$\frac{\partial v}{\partial \nu} = \sum_{i,j} \alpha_{ij} \frac{\partial v}{\partial x_i} \frac{\partial n}{\partial x_j}.$$

Finally, the vectors b and β will be used to represent $(b_1, \dots, b_n)$ and $(\beta_1, \dots, \beta_n)$, respectively.

As in [2] and [3], our criteria will involve a nonnegative function $g_1(x)$ which assures that the quadratic form

$$Q_1(\xi) \equiv \sum_{i,j=1}^n \alpha_{ij} \xi_i \xi_j + 2\xi_{n+1} \sum_{i=1}^n \beta_i \xi_i + g_1 \xi_{n+1}^2$$

be nonnegative definite. Since (2) is elliptic, such a function $g_1(x)$ can be determined by the condition that the determinant

$$\begin{vmatrix} \alpha_{11} & \cdots & \alpha_{1n} & \beta_1 \\ \vdots & & \vdots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} & \beta_n \\ \beta_1 & \cdots & \beta_n & g_1 \end{vmatrix}$$

be nonnegative. In addition we shall introduce a differentiable vector valued function $p(x) = (p_1(x), \dots, p_n(x))$ and determine $g_2(p, x)$ by the condition that

$$Q_2(\eta, p) \equiv \sum_{i,j=1}^n (a_{ij} - \alpha_{ij}) \eta_i \eta_j + 2\eta_{n+1} \sum_{i=1}^n (b_i - \beta_i - p_i) \eta_i + g_2 \eta_{n+1}^2$$

be nonnegative definite. The existence of such a $g_2(p, x)$ clearly requires that $\sum (a_{ij} - \alpha_{ij}) \xi_i \xi_j$ be nonnegative definite, and in case this form is positive definite, $g_2(p, x)$ can also be obtained by requiring that an appropriate determinant be nonnegative. One must also choose p such that $p - b + \beta$ lies in the range of the matrix $(a_{ij}) - (\alpha_{ij})$; g_2 can then be chosen to be $(p - b + \beta)^* [(a_{ij})^{-1} - (\alpha_{ij})^{-1}] (p - b + \beta)$.

THEOREM 1. Let $u(x)$ and $v(x)$ be nontrivial solutions of (1) and (2), respectively, satisfying

$$\frac{\partial u}{\partial n} + s(x)u = 0 \quad \text{on } \partial D, \quad \frac{\partial v}{\partial \nu} + \sigma(x)v = 0 \quad \text{on } \partial D.$$

If there exists a differentiable vector function $\mathbf{p}(x)$ such that²

$$\begin{aligned} (3) \quad & \int_D \left[\sum_{i,j} (a_{ij} - \alpha_{ij}) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + 2u \nabla u \cdot (\mathbf{b} - \boldsymbol{\beta} + \mathbf{p}) \right. \\ & \left. + (c - \gamma - g_1 + \nabla \cdot \mathbf{p}) u^2 \right] dx \\ & + \int_{\partial D} (s - \sigma - \mathbf{p} \cdot \mathbf{n}) u^2 dS > 0, \end{aligned}$$

then $v(x)$ has a zero in $\bar{D}$.

PROOF. Suppose $v(x) \neq 0$ in $\bar{D}$. Then the following generalized Picone identity established in [3] is valid.

$$\begin{aligned} (4) \quad & \sum_j \frac{\partial}{\partial x_j} \left[\frac{u}{v} \left(v \sum_i a_{ij} \frac{\partial u}{\partial x_i} - u \sum_i \alpha_{ij} \frac{\partial v}{\partial x_i} \right) \right] \\ & = (c - \gamma - g_1) u^2 + \sum_{i,j} \alpha_{ij} \left(\frac{\partial u}{\partial x_i} - \frac{u}{v} \frac{\partial v}{\partial x_i} \right) \left(\frac{\partial u}{\partial x_j} - \frac{u}{v} \frac{\partial v}{\partial x_j} \right) \\ & + 2u \sum_i \beta_i \left(\frac{\partial u}{\partial x_i} - \frac{u}{v} \frac{\partial v}{\partial x_i} \right) + g_1 u^2 \\ & + \sum_{i,j} (a_{ij} - \alpha_{ij}) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + 2u \sum (b_i - \beta_i) \frac{\partial u}{\partial x_i}. \end{aligned}$$

Motivated by a device of Picard, we also note that for any differentiable vector function $\mathbf{p}(x)$,

$$(5) \quad \nabla \cdot (u^2 \mathbf{p}) \equiv 2u \nabla \cdot \mathbf{p} + u^2 \nabla \cdot \mathbf{p}.$$

Adding the above identities yields

² In case $u(x) = 0$ on part or all of ∂D , this portion of ∂D is to be excluded from the surface integral in (3).

$$\begin{aligned}
 \sum_j \frac{\partial}{\partial x_j} \left[\frac{u}{v} \left(v \sum_i a_{ij} \frac{\partial u}{\partial x_i} - u \sum_i \alpha_{ij} \frac{\partial v}{\partial x_i} \right) + u^2 p_j(x) \right] \\
 (6) \qquad \qquad \qquad = \sum_{i,j} (a_{ij} - \alpha_{ij}) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + 2u \nabla u \cdot (b - \beta + p) \\
 \qquad \qquad \qquad + (c - \gamma - g_1 + \nabla \cdot p) u^2 + Q_1(\xi)
 \end{aligned}$$

where $\xi_i = \partial u / \partial x_i - (u/v)(\partial v / \partial x_i)$; $i = 1, \dots, n$, and $\xi_{n+1} = u$.

Integrating both sides of (6) over D , applying Green's theorem, and making use of the prescribed boundary behavior of $u(x)$ and $v(x)$ yields

$$\begin{aligned}
 \int_{\partial D} (\sigma - s + p \cdot n) u^2 dS \\
 = \int_D \left[\sum_{i,j} (a_{ij} - \alpha_{ij}) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + 2u \nabla u \cdot (b - \beta + p) \right. \\
 \qquad \qquad \qquad \left. + (c - \gamma - g_1 + \nabla \cdot p) u^2 + Q_1(\xi) \right] dx.
 \end{aligned}$$

However, this identity clearly contradicts hypothesis (3) and shows that $v(x) = 0$ for some $x \in \overline{D}$.

REMARKS:

1. In case ∂D is of bounded curvature, the conclusion of Theorem 1 can be strengthened to read "then $v(x)$ has a zero in D " (see [6]).

2. The techniques used to establish Theorem 1 also lend themselves to generalizations such as those considered by Swanson in [7] and [8]. In particular the coefficients of (1) and (2) may depend on $u(x)$ and $v(x)$, equations (1) and (2) may be replaced by certain differential inequalities, D may be replaced by an unbounded domain, and complex valued solutions may be considered.

Considering the special case where $u(x) \equiv 0$ on ∂D eliminates the surface integral from condition (3) in Theorem 1. The choice $p(x) = \beta(x) - b(x)$ then leads to the following comparison theorem of Swanson [2].

COROLLARY 1. *Let $u(x)$ be a nontrivial solution of (1) satisfying $u = 0$ on ∂D . If*

$$(3') \quad \int_D \left[\sum_{i,j} (a_{ij} - \alpha_{ij}) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + (c - \gamma - g_1 + \nabla \cdot (\beta - b)) u^2 \right] dx > 0$$

then every solution $v(x)$ of (2) has a zero in $\overline{D}$.

Another comparison theorem due to the author [3] is obtained by choosing $p(x) \equiv 0$ and noting that the terms of (3) which involve derivatives are bounded below by $-g_2 u^2$.

COROLLARY 2. *Suppose $u(x)$ and $v(x)$ are nontrivial solutions of (1) and (2), respectively, and that*

$$\int_D (c - \gamma - g_1 - g_2) u^2 dx + \int_{\partial D} (s - \sigma) u^2 dx > 0.$$

Then $v(x)$ has a zero in $\bar{D}$.

Another context in which the results of [2] and [3] differ is that the technique of [3] allows generalization to solutions of (1) and (2) satisfying mixed boundary conditions on ∂D , while the technique of [2] seems to be limited to the case where $u(x) = 0$ on ∂D . However, the choice $p(x) \equiv g(x) - b(x)$ in Theorem 1 yields a generalization of the comparison theorem of [2] to mixed boundary conditions.

It should also be observed that when $u = 0$ on ∂D , (3) is independent of the choice of $p(x)$. However, in applying these techniques to specific equations, the inclusion of $p(x)$ in (3) is a convenience, even in the case $u = 0$ on ∂D .

In case $n = 1$, (1) and (2) become

$$(1') \quad -\frac{d}{dx} \left(a(x) \frac{du}{dx} \right) + 2b(x) \frac{du}{dx} + c(x)u = 0 \text{ and}$$

$$(2') \quad -\frac{d}{dx} \left(\alpha(x) \frac{dv}{dx} \right) + 2\beta(x) \frac{dv}{dx} + \gamma(x)v = 0$$

respectively. Here $g_1(x)$ and $g_2(p, x)$ can be found explicitly. It follows readily from the definitions that the most propitious choices are³

$$g_1(x) = \beta(x)^2 / \alpha(x) \text{ and} \\ g_2(p, x) = \frac{(b(x) - \beta(x) + p(x))^2}{a(x) - \alpha(x)}.$$

Thus for ordinary differential equations Theorem 1 implies the following.

THEOREM 2. *Let $u(x)$ be a nontrivial solution of (1') satisfying $u(x_1) = u(x_2) = 0$. If there exists a differentiable function $p(x)$ such that*

³ Implicit in the definition of g_2 is the assumption that $a(x) - \alpha(x) > 0$ except where the numerator vanishes.

$$(7) \quad \int_{x_1}^{x_2} [(a-\alpha)u'^2 + 2(b-\beta+p)uu' + (c-\gamma-\beta^2/\alpha+p')u^2] dx > 0,$$

then every solution $v(x)$ of (2') has a zero in (x_1, x_2) .

The corollaries allow analogous formulation. In particular, Corollary 2 implies the following.

COROLLARY 3. Let $u(x)$ be a nontrivial solution of (1') on (x_1, x_2) satisfying $u(x_1) = u(x_2) = 0$. If

$$\int_{x_1}^{x_2} \left(c - \gamma - \frac{\beta^2}{\alpha} - \frac{(b-\beta)^2}{a-\alpha} \right) u^2 dx > 0,$$

then every solution $v(x)$ of (2') has a zero in (x_1, x_2) .

As an example of a class of differential equations for which a choice of p other than $p(x) \equiv 0$ or $p(x) = \beta(x) - b(x)$ is desirable, consider

$$-(au')' - x^{-1}u' + cu = 0, \quad -(x^{2k}v')' + x^{k-1}v' + \gamma v = 0,$$

where k is any real number and $a(x)$ is of class C' and satisfies $a(x) > x^{2k}$. Choosing $p(x) = -x^{-1}$, (7) becomes

$$\int_{x_1}^{x_2} \left[c - \gamma - \frac{(1+x^k)^2}{x^2(a-x^{2k})} \right] dx > 0,$$

which is always preferable to the choice $p(x) \equiv 0$. In many cases the above choice of $p(x)$ is also preferable to $p(x) = b(x) - \beta(x) = -x^{-1} - x^{k-1}$.

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UNIVERSITY OF CALIFORNIA, DAVIS, CALIFORNIA 95616