RESTRICTIONS OF L^p TRANSFORMS

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ABSTRACT. Let G be a locally compact abelian group with dual Γ , E a subset of Γ , and ϕ a complex-valued function defined on Γ . Assume ϕ has σ -compact support. In this paper we prove that ϕ is a multiplier of type $(L^1, L^{p_1} \cap L^{p_2}, E)$ ($1 \leq p_1 \leq 2$, $1 < p_2 \leq \infty$) if and only if $\phi = \hat{f}$ a.e. on E for some $f \in L^{p_1}(G) \cap L^{p_2}(G)$. We give applications of this result to the problems of restrictions, uniqueness, inversion and characterization of L^p transforms.

Throughout the paper $M(G)$ denotes the measure algebra of G , $L^p(G)$ ($1 \leq p \leq \infty$) the usual Lebesgue space of index p formed with respect to Haar measure, $C(G)$ the set of all bounded continuous complex-valued functions on G , and $C_0(G)$ those functions in $C(G)$ which vanish at infinity.

The Fourier transform $\hat{f}$ of a function $f \in L^1(G)$ is defined by

$$\hat{f}(\gamma) = \int_G f(x)(-x, \gamma) dx \quad (\gamma \in \Gamma).$$

The Fourier-Stieltjes transform $\hat{\mu}$ of a measure $\mu \in M(G)$ is defined by

$$\hat{\mu}(\gamma) = \int_G (-x, \gamma) d\mu(x) \quad (\gamma \in \Gamma).$$

For $1 < p \leq 2$ the Fourier transform $\hat{f}$ of a function $f \in L^p(G)$ may also be defined (see [3, p. 152]) and we have that $\hat{f} \in L^q(\Gamma)$ where $1/p + 1/q = 1$. In all cases (i.e., measures and L^p functions) the Fourier transformation is one-to-one. We fix the Haar measure on G so that the Fourier transformation maps $L^2(G)$ isometrically onto $L^2(\Gamma)$.

Let E be a subset (not necessarily measurable) of Γ and ϕ a complex-valued function defined on Γ . Suppose further that ϕ has σ -compact support. ϕ is said to be a multiplier of type $(L^1, L^{p_1} \cap L^{p_2}, E)$ ($1 \leq p_1 \leq 2$, $1 < p_2 \leq \infty$) if given $f \in L^1(G)$, there corresponds a $g \in L^{p_1}(G) \cap L^{p_2}(G)$ such that $\phi \hat{f} = \hat{g}$ a.e. on E . The

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set of all multipliers of type $(L^1, L^{p_1} \cap L^{p_2}, E)$ will be denoted by $(L^1, L^{p_1} \cap L^{p_2}, E)$ and $(L^1, L^{p_1} \cap L^{p_2}, \Gamma)$ by $(L^1, L^{p_1} \cap L^{p_2})$. The multiplier classes, $(L^1, L^1 \cap C, E)$, $(L^1, L^1 \cap C_0, E)$, etc., are defined analogously.

The multiplier problem $(L^1, L^{p_1} \cap L^{p_2})$ has been studied in special cases by Edwards [2, p. 368] and Ryan [6, p. 110]. We prove the following general theorem:

THEOREM 1. $(L^1, L^{p_1} \cap L^{p_2}, E) = (L^{p_1}(G) \cap L^{p_2}(G))^\wedge | E$, $1 \leq p_1 \leq 2$ and $1 < p_2 \leq \infty$.

PROOF. By $(L^{p_1}(G) \cap L^{p_2}(G))^\wedge | E$ we mean the set of a.e. restrictions to E of Fourier transforms of functions belonging to $L^{p_1}(G) \cap L^{p_2}(G)$. If $\phi \in (L^{p_1}(G) \cap L^{p_2}(G))^\wedge | E$ then there is an $f \in L^{p_1}(G) \cap L^{p_2}(G)$ such that $\phi = \hat{f}$ a.e. on E . Let $g \in L^1(G)$ and consider the convolution $f * g$. Clearly $f * g \in L^{p_1}(G) \cap L^{p_2}(G)$ hence $\hat{f}\hat{g} \in (L^{p_1}(G) \cap L^{p_2}(G))^\wedge$ and this implies $\phi \in (L^1, L^{p_1} \cap L^{p_2}, E)$.

To prove the converse we adapt a method of Wells (see [7, p. 245]). Assume $\phi \in (L^1, L^{p_1} \cap L^{p_2}, E)$ and define $\mathfrak{A}$ as follows:

$$\mathfrak{A} = \{f \in L^{p_1}(G) \cap L^{p_2}(G) : \hat{f} = 0 \text{ a.e. on } E\}.$$

We equip $L^{p_1}(G) \cap L^{p_2}(G)$ with the Banach space norm $\|f\| = \|f\|_{p_1} + \|f\|_{p_2}$. Then $\mathfrak{A}$ is a closed linear subspace of $L^{p_1}(G) \cap L^{p_2}(G)$. Hence the quotient space $L^{p_1}(G) \cap L^{p_2}(G) / \mathfrak{A}$ is a Banach space with the usual quotient norm.

The multiplier function ϕ induces a linear mapping T_ϕ from $L^1(G)$ into $L^{p_1}(G) \cap L^{p_2}(G) / \mathfrak{A}$ as follows:

$$T_\phi(f) = g + \mathfrak{A} \quad \text{where } \phi\hat{f} = \hat{g} \text{ a.e. on } E.$$

By the closed graph theorem there is a constant $K > 0$ such that given $f \in L^1(G)$ there corresponds a $g \in L^{p_1}(G) \cap L^{p_2}(G)$ where

$$\|g\|_{p_1} + \|g\|_{p_2} \leq K\|f\|_1 \quad \text{and} \quad \phi\hat{f} = \hat{g} \text{ a.e. on } E.$$

Let k_n be any sequence of kernels for ϕ . This means that $k_n \in L^1(G)$, $\|k_n\|_1 \leq 2$, $\hat{k}_n$ has compact support and $\hat{k}_n\phi \rightarrow \phi$ a.e. Such kernels exist by hypothesis (see Theorem 2.6.8 of [5, p. 53]). If $E = \Gamma$ then by Theorem 1 of [2, p. 368] ϕ has σ -compact support. Hence, there corresponds $h_n \in L^{p_1}(G) \cap L^{p_2}(G)$ where

$$\hat{h}_n = \phi\hat{k}_n \text{ a.e. on } E \quad \text{and} \quad \|h_n\|_{p_1} + \|h_n\|_{p_2} \leq 2K.$$

Assume temporarily that $1 < p_1 \leq 2$ and $1 \leq p_2 \leq \infty$. We may then assume without loss of generality that $h_n \rightarrow h$ weakly in $L^{p_1}(G)$. By

a theorem of Mazur [8, p. 120] there is a sequence of convex linear combinations $r_n = \sum_k \alpha_{nk} h_k$ such that $r_n \rightarrow h$ strongly in $L^{p_1}(G)$.

Moreover, in forming r_n we may assume $k \geq n+1$. This means that $\alpha_{nk} = 0$ if $k \leq n$; hence $\lim_{n \rightarrow \infty} \alpha_{nk} = 0$. Thus (α_{nk}) is a regular Toeplitz summation matrix.

Without loss of generality we may assume $r_n \rightarrow \hat{h}$ a.e. and $\hat{h}_n \rightarrow \phi$ a.e. on E . By the regularity of (α_{nk}) , $\hat{h} = \phi$ a.e. on E . We may also assume without loss of generality that $r_n \rightarrow h$ a.e. Since $\|r_n\|_{p_2} \leq 2K$ we may conclude by Fatou's lemma or uniform boundedness that $h \in L^{p_2}(G)$.

Suppose $p_1 = 1$ and $p_2 > 2$. Then the fact that $\|h_n\|_1 + \|h_n\|_{p_2} \leq 2K$ implies that $\|h_n\|_2 \leq 2K$. We may assume without loss of generality that $h_n \rightarrow h$ weakly in $L^2(G)$. As before we have a sequence $r_n = \sum_k \alpha_{nk} h_k$ such that $r_n \rightarrow h$ in $L^2(G)$ and (α_{nk}) is a regular Toeplitz summation matrix. By the regularity of (α_{nk}) , $\hat{h} = \phi$ a.e. on E .

We may also assume without loss of generality that $r_n \rightarrow h$ a.e. Since $\|r_n\|_1 \leq 2K$ and $\|r_n\|_{p_2} \leq 2K$, Fatou's lemma or uniform boundedness implies that $h \in L^1(G) \cap L^{p_2}(G)$. This concludes the proof.

An immediate consequence of Theorem 1 is the following corollary.

COROLLARY. $(L^1, L^1 \cap C_0, E) = (L^1, L^1 \cap C, E) = (L^1(G) \cap L^\infty(G))^\wedge \upharpoonright E$.

We now give some applications of Theorem 1 which are of interest for noncompact LCA groups.

THEOREM 2. Suppose $\hat{f} = \hat{\mu}$ a.e. on E where $\mu \in M(G)$ and $f \in L^p(G)$, $1 \leq p \leq 2$. Then there is an $h \in L^1(G) \cap L^p(G)$ such that $\hat{h} = \hat{\mu} = \hat{f}$ a.e. on E .

PROOF. Compare this with Theorem 1 of [1, p. 258] and Theorem 7 of [6, p. 108]. The proof of the present theorem follows from the observation that $\hat{f}$ is a multiplier of type $(L^1, L^1 \cap L^p, E)$.

THEOREM 3. Suppose $\hat{f}_1 = \hat{f}_2$ a.e. on E where $f_1 \in L^{p_1}(G)$ and $f_2 \in L^{p_2}(G)$, $1 \leq p_1 \leq 2$ and $1 \leq p_2 \leq 2$. Then there is an $h \in L^{p_1}(G) \cap L^{p_2}(G)$ such that $\hat{h} = \hat{f}_1 = \hat{f}_2$ a.e. on E .

PROOF. $\hat{f}_1$ and $\hat{f}_2$ are multipliers of type $(L^1, L^{p_1} \cap L^{p_2}, E)$.

THEOREM 4. Suppose $f \in L^p(G)$, $1 \leq p \leq 2$ and $\hat{f} \in L^1(\Gamma)$. Then

$$f(x) = \int_{\Gamma} \hat{f}(\gamma)(x, \gamma) d\gamma \quad \text{a.e.}$$

PROOF. We may assume $1 < p \leq 2$ or else this is the classical in-

version theorem [3, p. 22]. Since $\hat{f} \in L^1(\Gamma)$ this implies $\hat{f} \in (L^1, L^p \cap L^2)$. Hence

$$f \in L^p(G) \cap L^2(G) \quad \text{and} \quad f(x) = \int_{\Gamma} \hat{f}(\gamma)(x, \gamma) d\gamma \quad \text{a.e.}$$

This last assertion follows from the fact that the Fourier transformation defined on $L^2(G)$ composed with the so-called inverse Fourier transformation on $L^2(\Gamma)$ is the identity map on $L^2(G)$.

THEOREM 5. *Let $g \in L^q(\Gamma)$ where $1/p + 1/q = 1$ and $1 < p \leq 2$. Let k_n be any sequence of kernels for g . Then $g = \hat{f}$ a.e. for some $f \in L^p(G)$ if and only if*

$$\sigma_n(x) = \int_{\Gamma} g(\gamma) \hat{k}_n(\gamma)(x, \gamma) d\gamma$$

is such that $\sigma_n \in L^p(G)$ and $\sup_n \|\sigma_n\|_p < \infty$.

PROOF. Recall that by a sequence of kernels k_n for g we mean $\|k_n\|_1 \leq 2$, $\hat{k}_n$ has compact support and $\hat{k}_n g \rightarrow g$ a.e. Suppose $g = \hat{f}$ a.e. for some $f \in L^p(G)$, $1 < p \leq 2$. Let k_n be a sequence of kernels for $\hat{f}$ and consider the convolution $f * k_n$. Note that $f * k_n \in L^p(G)$ and $(f * k_n)^{\wedge} = \hat{f} \hat{k}_n \in L^1(\Gamma)$. By Theorem 4

$$(f * k_n)(x) = \int_{\Gamma} \hat{f}(\gamma) \hat{k}_n(\gamma)(x, \gamma) d\gamma \quad \text{a.e.}$$

so

$$\|\sigma_n\|_p \leq \|f\|_p \|\hat{k}_n\|_1 \leq 2\|f\|_p$$

and we conclude that $\sup_n \|\sigma_n\|_p < \infty$.

Next assume $\sup_n \|\sigma_n\|_p < \infty$, where

$$\sigma_n(x) = \int_{\Gamma} g(\gamma) \hat{k}_n(\gamma)(x, \gamma) d\gamma.$$

Without loss of generality assume $\sigma_n \rightarrow f$ weakly in $L^p(G)$.

We observe that $g \hat{k}_n \in L^1(\Gamma)$ and this implies $\sigma_n \in C_0(G)$. Hence $\sigma_n \in L^2(G)$ and we conclude that $\hat{\sigma}_n \in L^2(\Gamma)$. Since the inverse Fourier transforms of $\hat{\sigma}_n$ and $g \hat{k}_n$ are both σ_n we have (by Theorem 3) $\hat{\sigma}_n = g \hat{k}_n$ a.e. Thus, $\hat{\sigma}_n \rightarrow g$ a.e.

Since $\sigma_n \rightarrow f$ weakly in $L^p(G)$ we form (as in Theorem 1) a sequence $r_n = \sum_k \alpha_{nk} \hat{\sigma}_k$ such that $r_n \rightarrow f$ in $L^p(G)$ and (α_{nk}) is a regular Toeplitz summation matrix. Assume without loss of generality that $r_n \rightarrow \hat{f}$ a.e.

and $\hat{\sigma}_n \rightarrow g$ a.e. By the regularity of (α_{nk}) , $\hat{f} = g$ a.e. This concludes the proof.

REMARKS. Theorems 2 and 3, for the case $E = \Gamma$ have been proven by Hewitt and Ross [4, pp. 231–233]. Our proofs are different from theirs and were discovered independently. For a more general inversion theorem than our Theorem 4 see [4, pp. 240–241]. Theorem 5 seems to be new even for $G = \mathbb{R}$ and k_n , the Fejér kernel.

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