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ANNIHILATORS OF MODULES WITH A FINITE FREE RESOLUTION

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ABSTRACT. Let A be a commutative ring and let E be an A-module with a finite free resolution (see below for definition). Extending results known previously for noetherian rings, it is shown that $\text{ann}(E)$ = annihilator of E is trivial if and only if the Euler characteristic of $E = \chi(E) > 0$; in addition, if $\chi(E) = 0$, $\text{ann}(E)$ is dense (i.e. $\text{ann}(\text{ann}(E)) = 0$). Also, a local ring is constructed with its maximal ideal with a finite free resolution but consisting exclusively of zero-divisors and thus, contrary to the noetherian case, one does not always have a nonzero divisor in $\text{ann}(E)$ if $\chi(E) = 0$. Finally, if E has a finite resolution by (f.g.) projective modules it turns out that $\text{ann}(\text{ann}(E))$ is generated by an idempotent element.

1. For a commutative ring A, an A-module E has, we recall, a finite free resolution if there is an exact sequence

$$(*) \qquad 0 \to F_n \to \cdots \to F_1 \xrightarrow{\alpha} F_0 \to E \to 0$$

where each F_i is A-free with a basis of cardinality $rk(F_i) < \infty$. The integer $\chi(E) = \sum_{i=0}^{n} (-1)^i rk(F_i)$ is independent of the resolution (see [**2**], which we shall use as reference) and it is called the Euler characteristic of E.

It has been shown that the positivity of $\chi(E)$ is closely related to the faithfulness of E as an A-module. In fact, Auslander and Buchsbaum [**1**, Proposition 6.2] proved that if A is noetherian:

(i) If $\chi(E) > 0$ then annihilator of $E = \text{ann}(E) = (0)$.

(ii) If $\chi(E) = 0$ then $\text{ann}(E)$ contains a nonzero divisor.

Kaplansky in [**2**, p. 141] asks whether the chain conditions are needed at all. The purpose of this note is to show that the techniques there suffice to prove that in general we have:

THEOREM. (a) *If* $\chi(E) > 0$ *then* $\text{ann}(E) = (0)$.

(b) *If* $\chi(E) = 0$ *then* $\text{ann}(\text{ann}(E)) = (0)$.

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An example of a local ring will show that (b) is best possible, i.e. that $\operatorname{ann}(E)$ does not contain always a nonzero divisor.

2. For the proof we recall some points from [**2**, p. 139]:

(i) If S is a multiplicative set in A, then $\chi(E_S)=\chi(E)$.

(ii) Let A be a local ring in which every finitely generated ideal has a nontrivial annihilator; then any A-module with a finite free resolution is actually free. In this case, for brevity, we shall say that A is a 0-ring.

(a) Assume $\chi(E)>0$ and $0\neq a\in\operatorname{ann}(E)$. Let $J=\operatorname{ann}(a)$ and let P be a prime ideal minimal over J. Then, localizing at P we get that $J_P\neq A_P$ is the annihilator of aA_P and so $a_P\neq 0$. Also, PA_P is minimal over J_P and thus every finitely generated ideal I of A_P has a power $I^n\subset J_P$ and so $I^n a_P=(0)$; it follows easily that A_P is a 0-ring. Thus E_P is free of rank $\chi(E)>0$ and so $\operatorname{ann}(E_P)=(0)=(\operatorname{ann}(E))_P\supset aA_P$, a contradiction. Hence $\operatorname{ann}(E)=(0)$.

(b) Let $F(E)$ be the 0th Fitting invariant of E, i.e. if α in (*) is an $r\times m$ matrix, $F(E)$ is the ideal of A generated by the $m\times m$ minors of α. We know that $(\operatorname{ann}(E))^m\subset F(E)\subset\operatorname{ann}(E)$. In particular we have $\operatorname{ann}(\operatorname{ann}(E))=(0)$ iff $\operatorname{ann}(F(E))=(0)$. $\operatorname{Ann}(F(E))$ is easier to work with since $F(E)$ is finitely generated and so its annihilator "localizes."

Assume then $0\neq a\in\operatorname{ann}(F(E))$ and write $J=\operatorname{ann}(a)$. Let P be a prime ideal minimal over J. $J_P\neq A_P$ implies $a_P\neq 0$ and A_P is a 0-ring. As $\chi(E_P)=\chi(E)=(0)$, $E_P=(0)$ and $F(E)_P=F(E_P)=A_P$, a contradiction again. This concludes the proof.

3. An ideal I with the property of $\operatorname{ann}(E)$ of (b) above, that is $\operatorname{ann}(I)=(0)$, is called dense. As it is well known, whether in general it contains a nonzero divisor depends on what the maximal rational extension of A looks like. We consider here an example of an ideal in a local ring, with projective dimension one, but in which every element is a zero divisor.

Let $R=k[[x,y]]$ = power series ring in x, y over the field k. Let M be the R-module $=\bigoplus\sum k(P)$, where $k(P)$ = field of quotients of R/P, P = prime ideal of height 1. Let $A=R\oplus M$, where addition is componentwise and multiplication given by the rule

$$(r,m)\cdot(s,n) = (rs,\, rn+sm).$$

Notice that M is an ideal of A, $M^2=(0)$ and that A is a local ring with maximal ideal P generated by x, y. Observe also that every element of P is a zero divisor. We claim that proj dim $P=1$. For this end, consider $A^2\to P\to 0$ with $(1,0)\to x$ and $(0,1)\to y$. Let us determine the module of relations L of P. Let $(a,b)\in L$; write $a=a_0+$

$\sum a_P$ (and similarly for b) where a_0 denotes the R-component of a and a_P its $k(P)$-component. Then

$$a_0x + b_0y = 0 \quad \text{and} \quad a_Px + b_Py = 0 \qquad \forall P.$$

Since x, y is a regular sequence in R, the first relation says that (a_0, b_0) is a unique R-multiple of $(y, -x)$. As for the other relations, since x, y cannot be both zero in $k(P)$ and this last is a field, (a_P, b_P) must be a unique $k(P)$-multiple of $(y, -x)$. Thus $L = A(y, -x)$ and proj dim $P = 1$.

4. Corollary. *Let A be a commutative ring and let*

$$0 \to E_n \to \cdots \to E_1 \to E_0 \to E \to 0$$

be a projective resolution of E where the E_i's are finitely generated. Then $\operatorname{ann}(\operatorname{ann}(E))$ *is generated by an idempotent element.*

Proof. For each prime ideal P, let $r(P) = \chi(E_P)$; since, for each i, $rk((E_i)_P)$ defines a continuous function from Spec $A \to \{\boldsymbol{Z} + \text{discrete topology}\}$, $r(P)$ is also a continuous function. Let $F(E)$ be the 0th Fitting invariant of E and write $J = \operatorname{ann}(F(E))$; then, for each prime P, $J_P = \operatorname{ann}(F(E_P))$ and is, by the Theorem, either A_P or (0) —depending on whether $r(P) > 0$—or $= 0$. This says that A/J is a flat A-module (cyclic) with support an open set of Spec A. According to [3, p. 506] A/J is A-projective and so J is generated by an idempotent. It is clear that $J = \operatorname{ann}(\operatorname{ann}(E))$.

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