AN l_1 EXTREMAL PROBLEM FOR POLYNOMIALS¹

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ABSTRACT. Let $\mathfrak{M}_n$ be the maximum of the l_1 norm, $\sum^n |c_k|$, of all n th degree polynomials satisfying $|\sum^n c_k z^k| \leq 1$ for $|z| = 1$. We prove that $\mathfrak{M}_n$ is asymptotic to $\sqrt{n}$, by exhibiting polynomials P_n (which are partial sums of certain Fourier series), whose l_1 norm is asymptotic to $\sqrt{n}$.

Let $\mathfrak{F}_n$ be the class of all n th degree polynomials satisfying $|\sum_{k=0}^n c_k z^k| \leq 1$ for $|z| = 1$. Consider the following maximum:

$$\mathfrak{M}_n = \max_{\{f\}} \sum_{k=0}^n |c_k| \quad (f(z) \equiv \sum c_k z^k \in \mathfrak{F}_n).$$

Using the Schwarz inequality, we have

$$\begin{aligned} \sum |c_k| &\leq ((n+1) \sum |c_k|^2)^{1/2} \\ (1) \quad &= (n+1)^{1/2} \left((2\pi)^{-1} \int_{-\pi}^{\pi} \left| \sum c_k e^{ikt} \right| dt \right)^{1/2} \leq (n+1)^{1/2}, \end{aligned}$$

so that $\mathfrak{M}_n \leq (n+1)^{1/2}$. Actually, it was shown by Newman that $\mathfrak{M}_n \leq \sqrt{n}$ for $n > 0$, equality holding for $n = 1, 2$, and 4 . H. S. Shapiro showed, moreover, that $n = 1, 2, 4$ are the only cases of equality.

A natural question to ask is, how close is $\mathfrak{M}_n$ to $\sqrt{n}$ as $n \rightarrow \infty$? It is the purpose of this paper to prove that $\mathfrak{M}_n/\sqrt{n} \rightarrow 1$. Let us note that if $\sum |c_k|$ were actually equal to $(n+1)^{1/2}$, then both inequalities in (1) would reduce to equality, which would mean that $|c_k|$ is constant, $k = 0, 1, \dots, n$, and $|\sum c_k e^{ikt}|$ is constant. Now, our present result tells us, roughly speaking, that there exist polynomials in $\mathfrak{F}_n$ for which $\sum |c_k|$ is close to $(n+1)^{1/2}$. This, in turn, means that $|c_k|$ and $|\sum c_k e^{ikt}|$ are close to constant.

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What happens if we insist that $|c_k|$ actually be constant? Let $\mathfrak{F}_n^*$ be the subclass of $\mathfrak{F}_n$ which has the additional restriction that $|c_k|$ is constant, $k=0, 1, \dots, n$. Let $\mathfrak{M}_n^* = \max_{\{f\}} \sum |c_k|$ ($f \in \mathfrak{F}_n^*$). Now it has long been known that $\mathfrak{M}_n^* \geq c\sqrt{n}$, $c > 0$ some absolute constant (cf. Newman [3]). From the work of Beller [1, Proposition 2] it follows that $\mathfrak{M}_n^* \geq (1.1717)^{-1}\sqrt{n}$, which seems to be the best constant exhibited thus far; the question of whether $\mathfrak{M}_n^*$ is asymptotic to $\sqrt{n}$, is unsolved. P. Erdős [2] conjectured that it is not. Translated into our terms, his conjecture states that $\mathfrak{M}_n^* \leq (1-A)\sqrt{n}$, where $A > 0$ is some absolute constant.

Another subclass of $\mathfrak{F}_n$ is $\mathfrak{G}_n$, the class of all n th degree polynomials with real coefficients satisfying $|\sum c_k z^k| \leq 1$ for $|z|=1$. Let $\mathfrak{R}_n = \max_{\{f\}} \sum |c_k|$ ($f \in \mathfrak{G}_n$). From the work of H. S. Shapiro [5] and W. Rudin [4] (see Newman [3]), it follows that $\mathfrak{R}_n \geq c\sqrt{n}$. The question of whether $\mathfrak{R}_n/\sqrt{n} \rightarrow 1$ remains unsettled.

In this paper we will prove something a bit stronger than $\mathfrak{M}_n/\sqrt{n} \rightarrow 1$, namely:

THEOREM. $\mathfrak{M}_n \geq \sqrt{n} - 3n^{3/10} \log n$, for sufficiently large n .

PROOF. For any natural number n , let m be defined as the positive solution of $n+1 = 2(m+5m^{4/5} \log m)$. Set

$$f_n(t) = \exp\{im/2\pi t^2 - (\pi^2 - t^2)^{-1}m^{-2/5}\} \quad (-\pi < t < \pi),$$

$$f_n(-\pi) = f_n(\pi) = 0.$$

Let c_k be the k th Fourier coefficient of $f_n(t)$. Set

$$P_n(z) = \sum_{k=-(n-1)/2}^{[(n+1)/2]} c_k z^k,$$

where $[x]$ is the greatest integer $\leq x$.

PROPOSITION. For sufficiently large n ,

- (i) $\sum_{k=-(n-1)/2}^{[(n+1)/2]} |c_k| \geq \sqrt{n} - 3n^{3/10} \log n$,
- (ii) $|P_n(e^{it})| \leq 1$.

The Theorem follows directly from the Proposition, since

$$\exp(i[\tfrac{1}{2}n]t)P_n(e^{it}) = \sum_{k=0}^n c_K e^{ikt}, \quad \text{where } K = k - [\tfrac{1}{2}n].$$

PROOF OF PROPOSITION, PART (i). Let a_k be the k th Fourier coefficient of $g(t) = \exp(im/2\pi t^2)$. Setting $u = (\sqrt{m}/\pi)t - k/\sqrt{m}$, we have

$$(2) \quad a_k = (2\pi)^{-1} \int_{-\pi}^{\pi} g(t) e^{-ik t} dt = \frac{1}{2} m^{-1/2} e^{i\alpha} \int_{-\sqrt{m}(1+k/m)}^{\sqrt{m}(1-k/m)} \exp(\tfrac{1}{2}\pi i u^2) du,$$

where $\alpha = -\pi k^2/2m$. Thus,

$$(3) \quad \begin{aligned} \left| |a_k| - (2m)^{-1/2} \right| &\leq \left| a_k - \tfrac{1}{2} m^{-1/2} (1+i) e^{i\alpha} \right| \\ &= \tfrac{1}{2} m^{-1/2} \left| \left\{ \int_{-\infty}^{-\sqrt{m}(1+k/m)} + \int_{\sqrt{m}(1-k/m)}^{\infty} \right\} \exp(\tfrac{1}{2}\pi i u^2) du \right|, \end{aligned}$$

where the equality follows from (2) and the identity $\int_0^\infty \exp(\tfrac{1}{2}\pi i u^2) du = (1+i)/2$.

Before proceeding, let us note that by making the change of variables $v^2 = u$ and integrating by parts, we get

$$\int_x^\infty \exp(\tfrac{1}{2}\pi i u^2) du = (\pi x)^{-1/2} i \cdot \exp(\tfrac{1}{2}\pi i x^2) + O(x^{-3/2}),$$

so that, for sufficiently large x ,

$$(4) \quad \left| \int_x^\infty \exp(\tfrac{1}{2}\pi i u^2) du \right| \leq 1/x.$$

Now, for $|k| \leq m - m^{3/4}$, we have $\sqrt{m}(1-k/m) \geq m^{1/4}$ and $\sqrt{m}(1+k/m) \geq m^{1/4}$. Thus, for sufficiently large m , and for k satisfying $|k| \leq m - m^{3/4}$, (3) and (4) together yield

$$\left| |a_k| - (2m)^{-1/2} \right| \leq \frac{1}{m(1 - (k/m)^2)} \leq (2m^{3/4} - m^{1/2})^{-1} \leq m^{-3/4},$$

so that $|a_k| \geq (2m)^{-1/2} - m^{-3/4}$ ($|k| \leq m - m^{3/4}$). Thus we have

$$(5) \quad \begin{aligned} \sum_{|k| \leq m - m^{3/4}} |a_k| &\geq (2m - 2m^{3/4} - 2)((2m)^{-1/2} - m^{-3/4}) \\ &\geq (2m)^{1/2} - 4m^{1/4}. \end{aligned}$$

We now show that $\sum |c_k|$ is not much smaller than $\sum |a_k|$. Let $d_k = a_k - c_k$. By the Parseval theorem,

$$\begin{aligned} \sum_{k=-\infty}^{\infty} |d_k|^2 &= (1/\pi) \int_0^\pi \left\{ 1 - \exp\left(\frac{-1}{(\pi^2 - t^2)m^{2/5}}\right) \right\}^2 dt \\ &< (1/\pi) \int_0^\pi \left\{ 1 - \exp\left(\frac{-1}{\pi(\pi - t)m^{2/5}}\right) \right\}^2 dt \\ &= (1/\pi) \int_0^\pi \left\{ 1 - \exp\left(\frac{-1}{\pi u m^{2/5}}\right) \right\}^2 du \end{aligned}$$

$$\begin{aligned}
&= \pi^{-2} m^{-2/5} \cdot \int_{\pi^{-2} m^{-2/5}}^{\infty} (1 - e^{-t})^2 t^{-2} dt \\
&< \pi^{-2} m^{-2/5} \int_0^{\infty} (1 - e^{-t})^2 t^{-2} dt < 2\pi^{-2} m^{-2/5}.
\end{aligned}$$

Thus, by the Schwarz inequality,

$$\begin{aligned}
\sum_{k=[-(n-1)/2]}^{[(n+1)/2]} |a_k - c_k| &= \sum |d_k| \leq \left((n+1) \sum |d_k|^2 \right)^{1/2} \\
&\leq \sqrt{2(m + 5m^{4/5} \log m)^{1/2}} \sqrt{2\pi^{-1} m^{-1/5}} \\
&\leq (2/\pi) m^{3/10} + (5/\pi) m^{1/10} \log m.
\end{aligned}$$

Combining this with (5), we obtain

$$(6) \quad \sum_{k=[-(n-1)/2]}^{[(n+1)/2]} |c_k| \geq (2m)^{1/2} - m^{3/10}$$

for sufficiently large m . But

$$\sqrt{n} \leq (2m)^{1/2} (1 + 5m^{-1/5} \log m)^{1/2} \leq (2m)^{1/2} + 2^{-1/2} 5m^{3/10} \log m,$$

so that $(2m)^{1/2} \geq \sqrt{n} - 2^{-1/2} 5m^{3/10} \log m$. Furthermore, $m^{3/10} \leq (n/2)^{3/10}$ and $\log m < \log n$. Applying these inequalities to (6), we obtain, for sufficiently large n ,

$$\sum_{k=[-(n-1)/2]}^{[(n+1)/2]} |c_k| \geq \sqrt{n} - 3n^{3/10} \log n. \quad \text{Q.E.D.}$$

PROOF OF PROPOSITION, PART (ii). Since $f_n(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt}$, we have

$$|P_n(e^{it})| \leq |f_n(t)| + \left| \left\{ \sum_{k=-\infty}^{[-(n+1)/2]} + \sum_{k=[(n+3)/2]}^{\infty} \right\} c_k e^{ikt} \right|.$$

Thus, it follows that

$$(7) \quad |P_n(e^{it})| \leq \exp(-\pi^{-2} m^{-2/5}) + \sum_{k=-\infty}^{[-(n+1)/2]} |c_k| + \sum_{k=[(n+3)/2]}^{\infty} |c_k|.$$

It is not hard to verify that $f_n(t)$ is C^∞ in $[-\pi, \pi]$ and that $f_n^{(k)}(-\pi) = f_n^{(k)}(\pi) = 0$ ($k = 0, 1, 2, \dots$). Thus, integrating by parts repeatedly, we have

$$c_j = (2\pi)^{-1} \int_{-\pi}^{\pi} f_n(t) e^{-ijt} dt = (2\pi)^{-1} (ij)^{-k} \int_{-\pi}^{\pi} f_n^{(k)}(t) e^{-ijt} dt$$

for any positive integer k , so that

$$(8) \quad |c_j| \leq |j|^{-k} \cdot \max_{\{-\pi \leq t \leq \pi\}} |f_n^{(k)}(t)| \quad (k = 0, 1, 2, \dots).$$

LEMMA 1. For $-\pi \leq t \leq \pi$,

$$|f_n^{(k)}(t)| \leq \left\{ m + 4\pi m^{2/5} k^2 \exp\left(\frac{-k}{2\pi m} - 1\right) \right\}^k \exp\left(\frac{e}{\pi^2 m^{2/5}} + \frac{k^2}{2\pi m}\right).$$

PROOF. We recall that $g(t) = \exp((im/2\pi)t^2)$. Set $h(t) = \exp(-(\pi^2 - t^2)^{-1}m^{-2/5})$, $-\pi < t < \pi$, $h(-\pi) = h(\pi) = 0$, so that $g(t)h(t) = f_n(t)$.

For $k = 0, 1, 2, \dots$,

$$(9) \quad g^{(k)}(t) = g(t) \sum_{j=0}^{[k/2]} \frac{1}{j!2^j} \prod_{p=0}^{2j-1} (k-p) \left(\frac{im}{\pi}\right)^{k-j} t^{k-2j}.$$

Indeed, it is true for $k = 0$, and if we assume the inductive hypothesis for $k-1$, then, differentiating both sides, we get equation (9), with the upper limit of the summation (seemingly) replaced by $[\frac{1}{2}(k+1)]$. But if k is even, $[\frac{1}{2}(k+1)] = [\frac{1}{2}k]$; if k is odd, $[\frac{1}{2}(k+1)] = \frac{1}{2}(k+1)$, and the $j = \frac{1}{2}(k+1)$ term vanishes, so that $\sum_{j=0}^{[(k+1)/2]} = \sum_{j=0}^{(k-1)/2}$.

From (9) it follows that, for $-\pi \leq t \leq \pi$,

$$|g^{(k)}(t)| \leq \sum_{j=0}^{[k/2]} \frac{k^{2j}}{j!2^j} m^{k-j} \pi^{-j} = m^k \sum_{j=0}^{[k/2]} (1/j!)(k^2/2\pi m)^j,$$

so that

$$(10) \quad |g^{(k)}(t)| \leq m^k \exp(k^2/2\pi m).$$

Now let $H(x) = e^{-1/x}$ ($x > 0$), $H(0) = 0$, so that $h(t) = H((\pi^2 - t^2)m^{2/5})$. As with (9), one can prove by induction that

$$(11) \quad h^{(k)}(t) = \sum_{j=0}^{[k/2]} \frac{1}{j!2^j} \prod_{p=0}^{2j-1} (k-p) (-2m^{2/5})^{k-j} t^{k-j} H^{(k-j)}((\pi^2 - t^2)m^{2/5}).$$

LEMMA 2. $|H^{(k)}(x)| \leq (2k^2/e)^k$ ($x \geq 0$; $k = 1, 2, \dots$).

PROOF. It is obvious that $H^{(k)}(0) = 0$ ($k = 1, 2, \dots$). Now for $x > 0$, $\epsilon > 0$, we have

$$(12) \quad H^{(k)}(x) = (k!/2\pi i) \int_C \frac{e^{-1/z}}{(z-x)^{k+1}} dz$$

where C is the circle $|z-x| = x/(1+\epsilon)$. If $\theta = \arg(z-x)$, then, for z on the circle,

$$\frac{1}{z} = \frac{1 + \cos \theta / (1 + \epsilon) - i \sin \theta / (1 + \epsilon)}{x(1 + 2 \cos \theta / (1 + \epsilon) + 1 / (1 + \epsilon)^2)},$$

so that

$$\begin{aligned} \operatorname{Re}(1/z) &= \frac{1 + \cos \theta / (1 + \epsilon)}{x(1 + 2 \cos \theta / (1 + \epsilon) + 1 / (1 + \epsilon)^2)} \\ &\geq \frac{1 + 1 / (1 + \epsilon)}{x(1 + 2 / (1 + \epsilon) + 1 / (1 + \epsilon)^2)} = \frac{1 + \epsilon}{x(2 + \epsilon)}. \end{aligned}$$

Applying this to (12), we obtain

$$|H^{(k)}(x)| \leq k!(x/(1+\epsilon))^{-k} \exp\left(-\frac{1+\epsilon}{x(2+\epsilon)}\right) = k!(2+\epsilon)^k y^k e^{-y},$$

where $y = (1+\epsilon)/(x(2+\epsilon))$. Therefore, $|H^{(k)}(x)| \leq k!((2+\epsilon)k/e)^k$. Since this holds for any $\epsilon > 0$, it follows that

$$(13) \quad |H^{(k)}(x)| \leq k!(2k/e)^k, \quad k = 1, 2, \dots$$

We now make use of the inequality

$$(14) \quad k! \leq e\sqrt{k}(k/e)^k, \quad k = 1, 2, \dots$$

Induction can be used to prove (14). It is true for $k=1$, and if $(k-1)! \leq e(k-1)^{1/2}((k-1)/e)^{k-1}$, then multiplying by k yields

$$k! \leq e\sqrt{k}k^k e^{1-k} (1 - 1/k)^{k-1/2}.$$

(14) now follows from the elementary inequality $(1 - 1/k)^{k-1/2} \leq e^{-1}$, $k=1, 2, \dots$

Inequality (13), when combined with (14), yields

$$|H^{(k)}(x)| \leq e\sqrt{k}(2k^2/e^2)^k \leq (2k^2/e)^k,$$

proving Lemma 2.

Applying Lemma 2 to (11), we obtain

$$\begin{aligned} |h^{(k)}(t)| &\leq \sum_{j=0}^{[k/2]} \frac{k^{2j}}{j!2^j} (2m^{2/5})^{k-j} \pi^{k-2j} (2(k-j)^2/e)^{k-j} \\ &\leq (4\pi m^{2/5})^k e^{-k} k^{2k} \sum_j (1/j!) e^j (4\pi^2 m^{2/5})^{-j} (k/(k-j))^{2j} \\ &\leq (4\pi m^{2/5} k^2/e)^k \sum_j (1/j!) (e\pi^{-2} m^{-2/5})^j. \end{aligned}$$

Thus, it follows that

$$(15) \quad |h^{(k)}(t)| \leq (4\pi m^{2/5} k^2/e)^k \exp(e\pi^{-2} m^{-2/5}).$$

Finally, recalling that $f_n(t) = g(t)h(t)$,

$$f_n^{(k)}(t) = \sum_{j=0}^k \binom{k}{j} g^{(k-j)}(t) h^{(j)}(t),$$

so by (10) and (15),

$$\begin{aligned} |f_n^{(k)}(t)| &\leq \exp(e\pi^{-2} m^{-2/5}) \cdot \sum_{j=0}^k \binom{k}{j} m^{k-j} \exp\left(\frac{(k-j)^2}{2\pi m}\right) (4\pi m^{2/5} j^2/e)^j \\ &\leq \exp(e\pi^{-2} m^{-2/5}) \sum_{j=0}^k \binom{k}{j} \left(m \cdot \exp\left(\frac{k}{2\pi m}\right)\right)^{k-j} (4\pi m^{2/5} j^2/e)^j \\ &= \exp\left(\frac{e}{\pi^2 m^{2/5}} + \frac{k^2}{2\pi m}\right) \left\{m + 4\pi m^{2/5} k^2 \exp\left(-\frac{k}{2\pi m} - 1\right)\right\}, \end{aligned}$$

proving Lemma 1.

Now, in (8), let $k = [m^{1/5}]$. Applying Lemma 1, we get

$$\begin{aligned} |c_j| &\leq |j|^{-[m^{1/5}]} \max |f_n^{([m^{1/5}])}(t)| \leq |j|^{(-m^{1/5}+1)} e(m + 4\pi m^{4/5} e^{-1}) m^{1/5} \\ &\leq |j|^{(-m^{1/5}+1)} m^{m^{1/5}} \exp(4\pi e^{-1} + 1) \leq 300 |j|^{(-m^{1/5}+1)} m^{m^{1/5}}, \end{aligned}$$

so that

$$\begin{aligned} \sum_{k=-\infty}^{[-(n+1)/2]} |c_k| + \sum_{k=[(n+3)/2]}^{\infty} |c_k| &\leq 300 m^{m^{1/5}} \left\{ \sum_{k=-\infty}^{[-(n+1)/2]} + \sum_{k=[(n+3)/2]}^{\infty} \right\} |k|^{-m^{1/5}+1} \\ &\leq 600 m^{m^{1/5}} \sum_{\substack{m+5m^{4/5} \\ \log m+1/2 \leq j < \infty}} k^{-m^{1/5}+1} \\ &\leq 600 m^{m^{1/5}} \int_{\substack{m+5m^{4/5} \\ \log m-1/2}}^{\infty} x^{-m^{1/5}+1} dx \\ &= 600 m^{m^{1/5}} (m^{1/5} - 2)^{-1} (m + 5m^{4/5} \log m - \frac{1}{2})^{-m^{1/5}+2} \\ &\leq 601 m^{9/5} (1 + 4m^{-1/5} \log m)^{-m^{1/5}} \leq 601 m^{9/5} 2^{-4 \log m} \\ &= 601 m^{-(4 \log 2 - 9/5)} \end{aligned}$$

for sufficiently large m .

Combining this with (7), we obtain

$$\begin{aligned} |P_n(e^{it})| &\leq \exp(-\pi^2 m^{-2/5}) + 601m^{-(4 \log 2 - 9/5)} \\ &< 1 - \pi^2 m^{-2/5} + \frac{1}{2}\pi^4 m^{-4/5} + 601m^{-(4 \log 2 - 9/5)} < 1 \end{aligned}$$

for sufficiently large m . Q.E.D.

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