COVERING ISOTOPIES OF M^{n-1} IN N^n

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ABSTRACT. We show that a continuous family of locally flat separating embeddings of an $(n-1)$ -manifold M^{n-1} into an n -manifold N^n , where the family is parametrized by a locally compact finite-dimensional metric space B , can be covered locally and sometimes globally by a continuous family of homeomorphisms of N^n onto itself, provided $n \neq 4$. Furthermore, the covering family can be chosen to extend a preassigned covering family corresponding to a compact connected subset of B . We derive a stronger result for embeddings of S^{n-1} in S^n , and show that the natural map from the space of orientation preserving homeomorphisms of S^n to the space of locally flat embeddings of S^{n-1} into S^n , $n \neq 4$, is a Serre fibration and a weak homotopy equivalence.

Černavskii has announced [2] a covering isotopy theorem for a manifold M^k in a manifold N^n , for $n \geq 5$ and $n-k \neq 2$. This author independently obtained a similar result for $n \neq 4$ and $n-k=1$. The two methods of proof are substantially different, and each theorem contains strong conclusions which the other fails to obtain.

Our result follows naturally from the local contractibility theorem of Edwards and Kirby [4], the isotopy extension theorem of Dyer and Hamstrom [3] based on the selection theorems of Michael [6], and a theorem of the author on extension of codimension one embeddings [8]. Given these results our proof is considerably simpler than Černavskii's proof; it does not, however, adapt to other dimensions.

THEOREM. *Let N^n , M^{n-1} be compact manifolds without boundary, $n \neq 4$; let M^{n-1} be a p.l. manifold. Let B be a locally compact finite-dimensional metric space, Z a compact connected subset of B , and $b_0 \in Z$. Let $\{h_b\}: M \rightarrow N$, $b \in B$, be a continuous family of locally flat embeddings which separate N , and let $\{G_b\}: N \rightarrow N$, $b \in Z$, be a continuous family of homeomorphisms such that $h_b = G_b h_{b_0}$ for all $b \in Z$. Then there is a neighborhood U of Z in B and a continuous family $\{H_b\}: N \rightarrow N$, $b \in U$, of homeomorphisms such that $H_b = G_b$ for all $b \in Z$ and $h_b = H_b h_{b_0}$ for all $b \in U$.*

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Furthermore, if B is a continuum and either

(1) $M = S^{n-1}$ and $N = S^n$, or

(2) Z is a weak deformation retract of B ,

then U may be taken to be the entire space B .

REMARKS. Previous theorems of this nature, with the exception of Černavskii's, carry a strong local triviality condition on the family $\{h_b\}$; we require only that each h_b be a locally flat embedding.

The chief disadvantage of our method is that we do not obtain weak local contractibility of the space of embeddings, as did Černavskii. The advantage is that the theorems of Michael [6] yield the covering homotopy property in situations where a fibration naturally appears (see Corollaries 2 and 3).

The restriction $n \neq 4$ is necessary only for Lemma 1. If this lemma is true for $n = 4$, so is the theorem.

COROLLARY 1. If N^n , M^{n-1} are as in the theorem and $n \neq 4$, and $h: M \times I^k \rightarrow N \times I^k$ is a k -isotopy such that for each $p \in I^k$ h_p is a locally flat embedding which separates N , then there is an ambient k -isotopy $H: N \times I^k \rightarrow N \times I^k$ such that for each $p \in I^k$, $h_p = H_p h_{p_0}$, where p_0 is a base point of I^k .

PROOF. In the theorem set $B = I^k$, $Z = \{p_0\}$, and $G_{p_0} = 1_N$. Since B deforms to Z the theorem applies.

Let $H(N)$ denote the space of homeomorphisms of N onto itself, and let $\text{Emb}(M, N)$ denote the space of locally flat embeddings of M into N . If M is a submanifold of N which separates N , and if $\text{Emb}_0(M, N)$ denotes those embeddings which are restrictions to M of elements of $H(N)$, then the restriction mapping $r: H(N) \rightarrow \text{Emb}_0(M, N)$ is onto. It follows from the theorem, with $B = I^k \times I$ and $Z = I^k \times \{0\}$, that r has the CHP for cubes. We thus obtain

COROLLARY 2. Let $r: H(N) \rightarrow \text{Emb}_0(M, N)$ be defined by $r(h) = h|_M$. If $n \neq 4$ and M separates N , then r is a Serre fibration.

Let $H^+(S^n)$ denote the orientation preserving homeomorphisms of S^n . By the generalized Schoenflies theorem of Brown, $\text{Emb}(S^{n-1}, S^n) = \text{Emb}_0(S^{n-1}, S^n)$. The fiber in this case is the space of orientation preserving homeomorphisms of S^n whose restriction to S^{n-1} is the identity, and this is contractible via the Alexander isotopy. From the homotopy exact sequence of the fibration we have

COROLLARY 3. If $n \neq 4$, the map $r: H^+(S^n) \rightarrow \text{Emb}(S^{n-1}, S^n)$ is a Serre fibration and a weak homotopy equivalence.

The ultimate result would be a section for the map r , that is, a canonical Schoenflies theorem in the purest sense. This does not follow from the theorem since $\text{Emb}(S^{n-1}, S^n)$ fails to satisfy the hypothesis for B .

LEMMA 1. *Let M^{n-1} be a closed p.l. manifold, and let $M^{n-1} \times R$ have the product metric. If $n \neq 4$ and $\epsilon > 0$, there is a $\delta > 0$ such that any locally flat embedding $h: M \times 0 \rightarrow M \times R$ satisfying $d(h, i) < \delta$ can be extended to a homeomorphism $H: M \times R \rightarrow M \times R$ such that $d(H, I) < \epsilon$.*

A version of Lemma 1 in the case $n \geq 5$ was first proved by this author in [8], for embeddings of S^{n-1} in $E^n - \{0\}$. Seebeck [7] generalized the technique to work for closed p.l. manifolds M . Since the proof is essentially contained in these two sources, we shall restrict ourselves to an outline here.

Let $n \geq 5$. For some $t_0 > 0$, it is possible to construct an embedding $H_0: M \times [0, t_0] \rightarrow M \times R$ with the following properties:

- (1) $H_0|_{M \times 0} = h$.
- (2) $H_0|_{M \times t_0}$ is a translation of $H_0|_{M \times 0}$ in the R -direction by the amount t_0 , i.e., if $H_0(x, 0) = (y, t)$, then $H_0(x, t_0) = (y, t + t_0)$.
- (3) H_0 is an ϵ -embedding (when δ is small).

The embedding H_0 is obtained by the following method. Choose $t_0 > 0$ such that the translate of $h(M \times 0)$ by the amount t_0 in the R -direction is disjoint from $h(M \times 0)$. If h is close to the inclusion embedding i , t_0 may be chosen small. Using the radial engulfing theorems of Bing [1] and the author [9], one constructs an embedding $F: M \times [0, t_0] \rightarrow M \times R$ such that:

- (i) $F(x, t_0) = h(x, 0)$, $x \in M$;
- (ii) $F(M \times t_0)$ and $F(M \times 0)$ are separated by $M \times t$ for some $t \in R$;
- (iii) the fiber $F(x \times [0, t_0])$ lies near the line $x \times R$ for each $x \in M$;
- (iv) the image of F lies on the "negative" side of $h(M \times 0)$ in $M \times R$.

Let A_1 denote the closed region between $h(M \times 0)$ and the translate $h(M \times 0) + t_0$, and set $A_2 = F(M \times [0, t_0])$. Push A_2 onto $A_1 \cup A_2$ using the separating level $M \times t$; then push $A_1 \cup A_2$ onto A_1 using the product structure of A_2 . The composition of these pushes is a homeomorphism $G: A_2 \rightarrow A_1$ which changes M -coordinates by only a small amount. The embedding $H_0 = GF$ is then the desired embedding. This procedure is described in detail in Lemma 3 of [8], for the case in which M is a sphere.

To get the homeomorphism H on all of $M \times R$, we simply stack these embeddings. On the set $M \times [mt_0, (m+1)t_0]$, for any integer m , H is obtained by translating $M \times [mt_0, (m+1)t_0]$ to $M \times [0, t_0]$, applying H_0 , and following with the inverse of the translation.

If $n = 3$, Lemma 1 follows from Corollary 3 of Kister [5]. If $n < 3$, the proof is trivial.

LEMMA 2. Let N^n be a manifold, $n \neq 4$, Q^n a compact manifold with boundary in $\text{Int } N$ and $M^{n-1} = \partial Q^n$. Suppose M is locally flat in N and admits a p.l. structure. Then for each $\epsilon > 0$ there is a $\delta > 0$ such that each locally flat embedding $h: M \rightarrow N$ with $d(h, i_M) < \delta$ can be extended to an embedding $H: Q \rightarrow N$ with $d(H, i_Q) < \epsilon$, such that H agrees with i_Q on some open subset of $\text{Int } Q$.

PROOF. Let $\phi: M \times R \rightarrow N$ be a bicollar on M . ϕ exists because M is locally flat and separates N . Let $U = \phi(M \times (-1, 1))$. Given $\eta > 0$, there is a $\delta > 0$ such that if $h: M \rightarrow N$ is locally flat and $d(h, i_M) < \delta$, then there is an embedding $\bar{h}: U \rightarrow N$ such that $d(\bar{h}, i_U) < \eta$, and $\bar{h}$ extends h . This follows from Lemma 1, since ϕ is uniformly continuous on $M \times (-1, 1) = \phi^{-1}(U)$. If $\eta > 0$ is sufficiently small, then by Theorem 5.1 of [4], there is a homeomorphism $g: U \rightarrow N$ such that $g|_M = i_M$, $g = \bar{h}$ outside some compact neighborhood of M in U , and $d(g, i_U) < \epsilon/2$. Define $H: Q \rightarrow N$ by the identity map on $Q - gU$ and by $\bar{h}g^{-1}$ on $Q \cap gU$. If in addition we choose $\eta < \epsilon/2$, then $d(\bar{h}, i_U) < \epsilon/2$, and so

$$d(\bar{h}g^{-1}, i_U) \leq d(\bar{h}, i_U) + d(g^{-1}, i_U) < \epsilon/2 + \epsilon/2 = \epsilon.$$

Then $d(H, i_Q) < \epsilon$, and since $H|_M = h$, the lemma is proved.

Let Q be a compact manifold with boundary. Let $H(Q, \partial Q)$ denote the space of homeomorphisms of Q onto itself whose restrictions to ∂Q are the identity.

LEMMA 3. $H(Q, \partial Q)$ is locally contractible.

This is Theorem 5.1 of [4]. (See the note following the statement of 5.1.)

A map $f: X \rightarrow Y$ between metric spaces is *completely regular* if given $y \in Y$ and $\epsilon > 0$ there is a $\delta > 0$ such that if $d(y', y) < \delta$ then there is an ϵ -homeomorphism of $f^{-1}(y)$ onto $f^{-1}(y')$.

The following lemma is a restatement of Theorem 2 of Dyer-Hamstrom [3], which in turn depends on a selection theorem of Michael [6].

LEMMA 4. Suppose X and Y are metric spaces, X complete and Y finite dimensional, and Q is a compact manifold with boundary. Suppose $f: X \rightarrow Y$ is a completely regular mapping such that:

- (1) for each $p \in Y$ there is a homeomorphism f_p of Q onto $f^{-1}(p)$;
- (2) there is a homeomorphism h of $\bigcup_p(\partial Q)$, $p \in Y$, onto $\partial Q \times Y$ such that the diagram

$$\begin{array}{ccc}
 \bigcup_{p \in Z} f_p(\partial Q) & \xrightarrow{h} & \partial Q \times Y \\
 & \searrow f & \downarrow \pi \\
 & & Y
 \end{array}$$

is commutative;

(3) for some closed subset Z of Y , there is a homeomorphism $g^*: \bigcup_{p \in Z} f_p(Q) \rightarrow Q \times Z$ which extends h in the diagram.

Then there is a neighborhood U of Z in Y and a homeomorphism h^* such that h^* extends h and g^* and the diagram

$$\begin{array}{ccc}
 \bigcup_{p \in U} f_p(Q) & \xrightarrow{h^*} & Q \times U \\
 & \searrow f & \downarrow \pi \\
 & & U
 \end{array}$$

is commutative. Furthermore, if $Q = B^n$ or if Z is a weak deformation retract of B , U may be taken to be the entire space Y .

The proof of Lemma 4 is adapted easily from the proof of Theorem 2 of Dyer-Hamstrom [3]. We give an outline here.

Let $h_p: \partial Q \rightarrow f_p(\partial Q)$ be defined by $h_p(x) = h^{-1}(x, p)$. For each $p \in Y$ let G_p denote the space of all homeomorphisms of Q onto $f^{-1}(p)$ which coincide with h_p on ∂Q . Let $G = \{G_p\}$, $p \in Y$, and let $G^* = \bigcup G$. Let $F: G^* \rightarrow Y$ be the function which maps each element of G_p to p .

To prove Lemma 4 we must find a section for F over some neighborhood U of Z which extends the existing section over Z induced by g^* . The necessary power comes from Michael; we use Dyer and Hamstrom's statement slightly augmented. (See Theorems 1.2 and 6.1 of [6].)

THEOREM M. *If A and Y are metric spaces such that A is complete and the covering dimension of $Y \leq n+1$, Z is a closed subset of Y , $F: A \rightarrow Y$ is a mapping such that the collection $\{F^{-1}(y)\}$, $y \in Y$, is lower semicontinuous and equi- LC^n , and $f: Z \rightarrow A$ is a map such that for $z \in Z$, $f(z) \in F^{-1}(z)$, then there is a neighborhood U of Z in Y such that f extends to a map $f^*: U \rightarrow A$ with $f(u) \in F^{-1}(u)$ for $u \in U$. If $\pi_i(F^{-1}(y)) = 0$ for all $y \in Y$ and all $i \leq n$, or if Z is a weak deformation retract of Y , then U may be taken to be the entire space Y .*

Now set $G^* = A$ in Theorem M. We must verify the following hypotheses:

(i) G^* is topologically complete. Let $\bar{G}_p$ denote the closure of G_p in the space of continuous functions of Q into $f^{-1}(p)$, let $\bar{G} = \{\bar{G}_p\}$, $p \in Y$, and let $\bar{G}^* = \bigcup \bar{G}$. By applying Lemma 2 we see that $\bar{G}^*$, with the supremum metric, is complete. But G^* is a G_δ subset of $\bar{G}^*$, so G^* is topologically complete.

(ii) The collection G is equi- LC^m for each integer m . That is, given $p \in Y$, $g \in G_p$, and $\epsilon > 0$, there is a $\delta > 0$ such that if $p' \in Y$ then any mapping of S^k , $k \leq m$, into $G_{p'} \cap N(g, \delta)$ is homotopically trivial in $G_{p'} \cap N(g, \epsilon)$. This fact follows immediately from Lemma 3 and the complete regularity of f .

(iii) The collection G is lower semicontinuous. This is also immediate from Lemma 2.

The function $F: G^* \rightarrow Y$ thus satisfies the hypotheses of Michael's theorem. We conclude that F admits a section on some neighborhood U of Z in Y , and this section induces the desired homeomorphism h^* .

Michael's theorem gives a global section whenever Q has the property that $\pi_i(H(Q, \partial Q))$ vanishes for $i < \dim Y$. But $\pi_i(H(B^n, S^{n-1}))$ vanishes for all i , so we may take $U = Y$ in this case.

We now prove the theorem.

Let $b_0 \in Z$. The embedding $h_{b_0}(M)$ separates N into a collection $Q_1, Q_2, \dots, Q_k$ of compact manifolds with boundary. Select one of these and call it Q . Identify M with $h_{b_0}(M)$, and h_{b_0} with $i_M: M \rightarrow N$. By Lemma 2, there is a neighborhood W of b_0 in B such that if $b \in \bar{W}$, then there is an extension $\tilde{h}_b: Q \rightarrow N$ of $h_b: M \rightarrow N$ which agrees with i_Q on some open subset of $\text{Int } Q$. This determines uniquely, for each $b \in \bar{W}$, a subset $Q_b = \tilde{h}_b(Q)$ of N which varies continuously (setwise) with b . Since B is locally compact, we shall choose W so that $\bar{W}$ is compact. Since Z is a continuum, the choice of Q_b thus extends uniquely to some compact neighborhood $\bar{U}$ of Z .

Set $X = \bigcup (Q_b \times b)$, $b \in \bar{U}$. Then X is complete since N and $\bar{U}$ are complete. Define $f: X \rightarrow \bar{U}$ by $f = \pi|_X$, where $\pi: N \times B \rightarrow B$ is the projection map. The map f is completely regular by Lemma 2. We have the commutative diagram

$$\begin{array}{ccc} \bigcup_{b \in \bar{U}} (h_b(M) \times b) & \xrightarrow{h^{-1}} & M \times \bar{U} \\ & \searrow f & \downarrow \pi \\ & & \bar{U} \end{array}$$

where $h: M \times B \rightarrow N \times B$ is given by hypothesis. By Lemma 4, there is a neighborhood V of Z in $\bar{U}$ and a homeomorphism h^* such that the diagram

$$\begin{array}{ccc}
 \bigcup_{b \in V} (Q_b \times b) & \xrightarrow{h^*} & Q \times V \\
 & \searrow f & \downarrow \pi \\
 & & V
 \end{array}$$

is commutative, and h^* extends h^{-1} and g^{*-1} , where g^* is induced by the family $\{G_b\}$.

Set $H = h^{*-1}$. Then $H: Q \times V \rightarrow N \times V$ has the property that $H_b|_M = h_b$ for all $b \in V$. Since h_{b_0} was identified with i_M , we have that $h_b = H_b h_{b_0}$ for all $b \in V$.

Repeat the argument using the other components of the complement of M in N . We then obtain, for each $i = 1, 2, \dots, k$, a neighborhood V_i of Z in B over which the embedding $h: M \times V_i \rightarrow N \times V_i$ can be covered by an embedding $H_i: Q_i \times V_i \rightarrow N \times V_i$. Set $V = \bigcap_1^k V_i$ and let $H: N \times V \rightarrow N \times V$ be obtained by glueing together the homeomorphisms H_i , $i = 1, \dots, k$. The induced family $\{H_b\}$, $b \in V$, satisfies the conclusion of the theorem.

If B is a continuum, then clearly the choice of Q_b extends uniquely to all of B . We may then set $X = \bigcup (Q_b \times b)$, $b \in B$, which is complete. If $M = S^{n-1}$ and $N = S^n$, then each Q_b is homeomorphic to B^n , and by Lemma 4 we may take each V_i , $i = 1, 2$, to be all of B . This completes the proof of the theorem.

ADDED IN PROOF. D. B. Gauld, in Proc. Amer. Math. Soc. **27** (1971), 603–612, has a result which is related to Corollary 3. Gauld shows that if one can select continuously for each element f of $\text{Emb}(S^{n-1}, S^n)$ an element $\tilde{f}$ of $\text{Emb}(S^{n-1} \times [-1, 1], S^n)$ with $\tilde{f}|_{S^{n-1} \times 0} = f$, then one can construct a section (a canonical Schoenflies theorem) for the map r .

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