ON THE TOPOLOGICAL COMPLETION OF M -SPACE PRODUCTS

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ABSTRACT. Using the continuum hypothesis, an example is given of two countably compact spaces X and Y such that $X \times Y$ is an M -space, but is not countably compact. If μX denotes the completion of X with respect to its finest uniformity, the above example shows that $\mu(X \times Y)$ is not necessarily equal to $\mu X \times \mu Y$, even though $X \times Y$ is an M -space.

It is known that the product of countably compact spaces need not be countably compact [7], and that the product of M -spaces need not be an M -space [4]. Since countably compact spaces are M -spaces, the question remains:

Q: If X and Y are countably compact and $X \times Y$ is an M -space, is $X \times Y$ countably compact?

In [5], [6], Morita defines μX to be the topological completion of a completely regular T_1 -space X with respect to its finest uniformity and proves that if $X \times Y$ is an M -space, then $\mu(X \times Y) = \mu X \times \mu Y$ if and only if $K \times L$ is countably compact whenever K and L are closed countably compact subsets of X and Y , respectively. In general, $\mu(X \times Y) \neq \mu X \times \mu Y$; however, in [6], Morita asks whether $\mu(X \times Y) = \mu X \times \mu Y$ holds if $X \times Y$ is an M -space. He shows that this is equivalent to Q.

Assuming the continuum hypothesis, spaces X and Y are constructed to provide a negative answer to Q.

Throughout, all spaces are assumed to be completely regular and T_1 ; the word countable will mean countably infinite; N is the set of positive integers with the discrete topology.

A space K is countably compact if and only if each countable discrete subset of K has an accumulation point. A space X is an M -space if and only if it admits a closed, continuous mapping into a metric space such that point inverses are countably compact.

Since the spaces to be constructed are subsets of βN , the following facts will be reviewed.

If M is a countable discrete subset of βN , the Stone-Čech compactification of N , then $\text{cl } M$ (the closure of M in βN) is homeomorphic to βN , and $M^* = \text{cl } M - M$ is homeomorphic to $N^* = \beta N - N$.

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If M_1 and M_2 are disjoint countable subsets of βN such that $M_1 \cup M_2$ is discrete, then $\text{cl } M_1 \cap \text{cl } M_2 = \emptyset$ and $M_1^* \cap M_2^* = \emptyset$.

If M_1 and M_2 are countable discrete subsets of βN , $x \in M_1^*$ and $y \in M_2^*$, then the type of x relative to M_1 is said to be the same as the type of y relative to M_2 (and written $\tau(x, M_1) = \tau(y, M_2)$) if there is a homeomorphism of $\text{cl } M_1$ onto $\text{cl } M_2$ taking x onto y .

A point $x \in M^*$ is a P -point in M^* if every G_δ in M^* is a neighborhood of x in M^* . Assuming the continuum hypothesis, W. Rudin [8] showed that there are 2^c P -points in N^* (and hence in M^* for any countable discrete $M \subset \beta N$).

Define an equivalence relation $\sim$ on the P -points in M^* by $x \sim y$ if and only if $\tau(x, M) = \tau(y, M)$. Since there are only c autohomeomorphisms on $\text{cl } M$, there are 2^c equivalence classes of P -points.

LEMMA 1 (FROLÍK [2]). If M_1 is a countable discrete subset of βN and $M_2 \subset M_1$, then $\tau(x, M_2) = \tau(x, M_1)$ for each $x \in M_2^*$.

LEMMA 2 (FROLÍK [3]). If M_1 and M_2 are countable discrete subsets of βN , then $M_3 = (M_1 \cap M_2) \cup (M_1^* \cap M_2) \cup (M_1 \cap M_2^*)$ is discrete, $\text{cl } M_3 = \text{cl } M_1 \cap \text{cl } M_2$, and $M_3^* = M_1^* \cap M_2^*$.

THEOREM. If M_1 and M_2 are countable discrete subsets of βN and x is a P -point in both M_1^* and M_2^* , then $x \in (M_1 \cap M_2)^*$ and $\tau(x, M_1) = \tau(x, M_2)$.

PROOF. Since $x \in M_1^* \cap M_2^*$, either $x \in (M_1 \cap M_2)^*$, $x \in (M_1^* \cap M_2)^*$, or $x \in (M_1 \cap M_2^*)^*$, by Lemma 2. However, x is a P -point in M_1^* and so cannot be a limit point of any countable subset of M_1^* . Thus $x \notin (M_1^* \cap M_2)^*$. Similarly, $x \notin (M_1 \cap M_2^*)^*$. Since $x \in (M_1 \cap M_2)^*$ and $M_1 \cap M_2 \subset M_i$, for $i=1, 2$, Lemma 1 implies that $\tau(x, M_1) = \tau(x, M_1 \cap M_2) = \tau(x, M_2)$.

Construction of X and Y . The following example is a modification of the example of Novák [7], simplified by Frolík [1], of two countably compact spaces whose product is not countably compact.

βN has 2^c countable discrete subsets which may be well ordered as $M_1, M_2, \dots, M_\alpha, \dots; \alpha < \aleph$, where $\aleph$ is the smallest ordinal of cardinality 2^c .

Let ϕ be a mapping from the countable discrete subsets of βN into N^* , defined as follows:

$\phi[M_1]$ is a P -point in M_1^* . For each α , $1 < \alpha < \aleph$, $\phi[M_\alpha]$ is a P -point in M_α^* such that $\tau(\phi[M_\alpha], M_\alpha) \neq \tau(\phi[M_\beta], M_\beta)$ for any $\beta < \alpha$. This is possible since there are 2^c equivalence classes of P -points in M_α^* and α has fewer than 2^c predecessors.

Inductively, define a sequence $X_0, X_1, \dots, X_\gamma, \dots; \gamma < \omega_1$ of subsets of βN such that $X_0 = N$ and for each $\gamma < \omega_1$, $X_\gamma = \bigcup_{\alpha < \gamma} X_\alpha \cup \{\phi[M] : M \text{ is a countable discrete } \subset \bigcup_{\alpha < \gamma} X_\alpha\}$. Let $X = \bigcup \{X_\gamma : \gamma < \omega_1\}$. Since each countable discrete subset of X is contained in X_γ for some $\gamma < \omega_1$, it has an accumulation point in $X_{\gamma+1} \subset X$. X is therefore countably compact. For each $\gamma < \omega_1$, $\text{card } X_\gamma \leq c \cdot c + (c \cdot c)^{\aleph_0} = c$, so $\text{card } X \leq c$.

Let $Y = N \cup (\beta N - X)$. For each countable discrete $A \subset Y$, $\text{card cl } A = 2^c$, so $A^* \cap Y \neq \emptyset$ and A has an accumulation point in Y . Thus Y is also countably compact.

$X \times Y$ is not countably compact. If $\Delta = \{(n, n) : n \in N\}$, then Δ is a countable discrete subset of $X \times Y$. Any accumulation point of Δ must be of the form (p, p) for $p \in N^*$. Since $X \cap Y = N$, the set Δ has no accumulation point in $X \times Y$.

$X \times Y - \Delta$ is countably compact. Let $\{(x_i, y_i)\}$ be a countable discrete subset of $X \times Y - \Delta$. If $x_i = x$ (or $y_i = y$) for an infinite number of indices, then $\{(x_i, y_i) : x_i = x\}$ is a countable discrete subset of $\{x\} \times Y$, which is countably compact. $\{(x_i, y_i) : x_i = x\}$ thus has an accumulation point in $\{x\} \times Y \subset X \times Y$. It may, then, be assumed that $x_i \neq x_j$ and $y_i \neq y_j$ if $i \neq j$. It may also be assumed (by choosing an appropriate subset) that $M_1 = \{x_i\}$ and $M_2 = \{y_i\}$ are discrete subsets of X and Y , respectively, and that $M_1 \cap M_2 = \emptyset$ (since $x_i \neq y_j$ for all $i = 1, 2, \dots$).

Let h be the homeomorphism of $\text{cl } M_1$ onto $\text{cl } M_2$ defined by $h(x_i) = y_i$ for all $x_i \in M_1$. The point $\phi[M_1]$ is in X and is in $\text{cl } M_1$. Since h is a homeomorphism, $h\phi[M_1] \in \text{cl } M_2$, $\tau(\phi[M_1], M_1) = \tau(h\phi[M_1], M_2)$ and $h\phi[M_1]$ is a P -point in M_2^* .

If $h\phi[M_1] \in X$, then $h\phi[M_1] = \phi[M]$ for some countable discrete $M \subset X$; that is, $h\phi[M_1]$ is a P -point in M^* . Since $h\phi[M_1]$ is a P -point in both M^* and M_2^* , it follows from the theorem that $h\phi[M_1] \in (M \cap M_2)^*$ and $\tau(h\phi[M_1], M) = \tau(h\phi[M_1], M_2)$.

By the definition of ϕ , the second relation gives $M = M_1$ (because $\tau(h\phi[M_1], M_2) = \tau(\phi[M_1], M_1)$). The first relation implies that $M \cap M_2 \neq \emptyset$ and hence $M_1 \cap M_2 \neq \emptyset$. This contradiction shows that $h\phi[M_1] \notin X$ and must therefore be in Y . Since h is a homeomorphism, $(\phi[M_1], h\phi[M_1])$ is an accumulation point of $\{(x_i, h(x_i))\} = \{(x_i, y_i)\}$ in $X \times Y - \Delta$.

$X \times Y$ is an M -space. The real-valued function f on $X \times Y$ defined by

$$\begin{aligned} f(x, y) &= n & \text{if } (x, y) &= (n, n), \\ &= 0 & \text{if } (x, y) &\notin \Delta, \end{aligned}$$

is continuous (since Δ is both open and closed in $X \times Y$), closed, and $f^{-1}(r)$ is countably compact for each real r .

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