ON THE FOURIER-ALGEBRA OF A LOCALLY COMPACT AMENABLE GROUP

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ABSTRACT. Let G be a locally compact amenable group. The elements of the Fourier-algebra $A(G)$ are characterized with the aid of the theorem of Schoenberg-Eberlein-Eymard.

In this paper an extension of the theorem of Schoenberg-Eberlein-Eymard is shown, which yields a characterization of the Fourier-algebra $A(G)$ of a locally compact amenable group. For abelian groups our result is due to Raouf Doss [2].

Let G be a locally compact group with left-invariant Haar measure dx and let $L^p(G)$, $p \geq 1$, be the Banach space of p -integrable complex-valued functions on G . By $M^1(G)$ we denote the convolution-algebra of the bounded Radon-measures on G . Let ρ be the left-regular representation and Σ the set of all (classes of) strongly continuous unitary representations of G .

It is known that every $\pi \in \Sigma$ in the Hilbert space $\mathfrak{H}_\pi$ can be lifted to a nondegenerate representation of $L^1(G)$ by

$$\pi(f) = \int_G f(x)\pi(x)dx.$$

If $\|\pi(f)\|$ denotes the norm of the operator $\pi(f)$ on the Hilbert space $\mathfrak{H}_\pi$ then for every nonempty subset $\Omega \subset \Sigma$ $\|f\|_\Omega = \sup_{\pi \in \Omega} \|\pi(f)\|$ defines a C^* -seminorm on $L^1(G)$. The $\|\cdot\|_\Omega$ -norm completion of $L^1(G)$ is called the C^* -group algebra $C^*(G)$ and the completion of $L^1(G)$ under $\|\cdot\|_\rho$ -norm is called the reduced C^* -group algebra $C_\rho^*(G)$.

It is known that G is amenable iff $\|f\|_\Sigma = \|f\|_\rho$ for all $f \in L^1(G)$. In this case $C_\rho^*(G)$ is isomorphic to $C^*(G)$ [1]. Let u be a continuous linear functional on $C^*(G)$. Since $\|f\|_1 \geq \|f\|_\Sigma$ for all $f \in L^1(G)$, u defines also a continuous linear functional on $L^1(G)$. Therefore the dual space $B(G) = C^*(G)'$ can be considered as a subspace of $L^\infty(G)$. In [1] Eymard showed that $B(G)$ consists of bounded continuous functions and that $B(G)$ is a commutative Banach algebra with identity under pointwise multiplication.

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By $K(G)$ denote the space of all continuous functions f on G with compact support $\text{supp}(f)$. Then the space

$$K * K = \left\{ \sum_{i=1}^n f_i * g_i \mid f_i, g_i \in K(G), n \in \mathbb{N} \right\}$$

is contained in $B_p(G)$ and its closure $A(G)$, "the Fourier-algebra of G " is a closed ideal in $B(G)$.

REMARK. If G is abelian and $\hat{G}$ the character group of G , then $B(G)$ is the algebra of all Fourier-Stieltjes-transforms $\hat{\mu}$ of bounded Radon-measures $\mu \in M^1(\hat{G})$ with $\|\hat{\mu}\| = \|\mu\|_1$ and $A(G)$ is exactly the algebra of all Fourier-transforms $\hat{f}$ with $f \in L^1(\hat{G})$ and $\|\hat{f}\| = \|f\|_1$.

Let $VN(G)$ be the Von Neumann-algebra of operators on $L^2(G)$ generated by $\rho(K(G))$. It is known that $VN(G)$ contains all operators $\rho(\mu)$, $\mu \in M^1(G)$, and in [1] Eymard showed that $VN(G)$ is isometrically isomorphic to the dual space $A'(G)$ of $A(G)$ by virtue of the isomorphism Φ ,

$$\Phi: VN(G) \rightarrow A'(G),$$

defined by $\Phi(T) = \varphi_T$ with

$$\varphi_T((f * g)^\vee) = (Tf \mid g) \quad \text{for } f, g \in K(G) \text{ and } \check{f}(x) = f(x^{-1}).$$

At this the ultraweak topology of $VN(G)$ corresponds to the weak topology of the duality $\sigma(A', A)$.

$VN(G)$ is a $B(G)$ -module, where the action of $B(G)$ on $VN(G)$ is defined as follows:

If $u \in B(G)$ and $T \in VN(G)$, then uT is the uniquely determined operator $VN(G)$, for which

$$\varphi_{uT}(v) = \varphi_T(uv) \quad \text{for all } v \in A(G).$$

We have $\|uT\|_p \leq \|u\| \|T\|_p$. Particularly, if $T = \rho(\mu)$ for a certain $\mu \in M(G)$, then $u\rho(\mu) = \rho(u\mu)$, where $u\mu$ is the usual product of the measure μ with the function u .

In [1, (2.3)(2.24)] Eymard generalized the theorem of Schoenberg-Eberlein from abelian groups to arbitrary locally compact groups:

Let $VN_D(G)$ be the set of all finite linear combinations $\delta = \rho(\sum c_i \delta_{x_i})$ of Dirac measures in $VN(G)$.

A complex valued function u defined on G is contained in $B(G)$ iff it is continuous and if there exists a constant $M \geq 0$ such that, for all $\delta \in VN_D(G)$,

$$\left| \sum c_i u(x_i) \right| \leq M \left\| \rho\left(\sum c_i \delta_{x_i}\right) \right\|_p, \\ \|u\| = \sup \left\{ \left| \sum c_i u(x_i) \right| ; \left\| \rho\left(\sum c_i \delta_{x_i}\right) \right\|_p \leq 1 \right\}.$$

With the aid of this theorem it is possible to determine when an element of $B(G)$ is actually in $A(G)$ in case G is amenable.

DEFINITION. Let $B_c(G)$ be the set of all functions $u \in B(G)$ such that for any $\epsilon > 0$ there exists a compact $K = K(\epsilon, u) \subset G$ such that $\|\rho(\sum c_i \delta_{x_i})\|_\rho \leq 1$ and $x_i \notin K$ implies $|\sum c_i u(x_i)| < \epsilon$.

THEOREM. If G is amenable, then $A(G) = B_c(G)$.

PROOF. For $\delta = \rho(\sum c_i \delta_{x_i}) \in VN_D(G)$ and $u \in A(G)$ we write $\sum c_i u(x_i) = \langle u | \delta \rangle$.

(1) Let $u \in A(G)$ and $\epsilon > 0$ be given. Since $A(G) \cap K(G)$ is dense in $A(G)$, there exists an $u_K \in A(G) \cap K(G)$ with compact support $K = \text{supp } u_K$ and $\|u - u_K\| < \epsilon$. Now let $\delta = \rho(\sum c_i \delta_{x_i})$, $\|\delta\|_\rho \leq 1$ where $x_i \notin K$. Then $|\langle u | \delta \rangle| = |\langle u - u_K | \delta \rangle| \leq \|u - u_K\| \|\delta\|_\rho \leq \epsilon$. Therefore $u \in B_c(G)$.

(2) For a measurable set V in G , let χ_V denote the characteristic function and $|V|$ the left Haar measure of V .

Let $u \in B_c(G)$ and $1 \geq \epsilon > 0$. By assumption there exists a compact $K \subset G$ such that for all $\delta \in VN_D(G)$ with $\text{supp } \delta \cap K = \emptyset$, one has $|\langle u | \delta \rangle| \leq \epsilon \|\delta\|_\rho$. Since G is amenable, there exists a compact $V \subset G$ with positive Haar measure such that (cf. [4]) $|KV|/|V| \leq 1 + \epsilon$.

Now consider the function

$$v = |V|^{-1} \chi_{KV} * \tilde{\chi}_V \quad \text{where } \tilde{\chi}_V(x) = \overline{\chi_V(x^{-1})}.$$

Then $v \in A(G)$ and $\|v\| \leq |V|^{-1} \|\chi_{KV}\|_2 \|\chi_V\|_2 \leq 1 + \epsilon$ and for all $x \in K$ one has $v(x) = 1$ and $\text{supp}(v) \subset KVV^{-1}$. For $K_1 = KVV^{-1}$ the same construction yields another function $w \in A(G)$ with the properties:

$$\|w\| \leq 1 + \epsilon \quad \text{and} \quad w(x) = 1 \quad \text{for all } x \in K_1.$$

Now we claim $\|u - wu\| \leq 9\epsilon$. To see this let $\delta \in VN_D(G)$ with $\|\delta\|_\rho \leq 1$. Then $\|v\delta\|_\rho \leq 1 + \epsilon$ and $\|(1-v)\delta\|_\rho \leq 2 + \epsilon$ and

$$|\langle u - wu | \delta \rangle| \leq |\langle u - wu | v\delta \rangle| + |\langle u - wu | (1-v)\delta \rangle|.$$

The first summand on the right side vanishes, because $\langle u - wu | v\delta \rangle = \langle (v - vw)u | \delta \rangle$ and $v = vw$ by choice of w . The second summand is dominated by

$$\begin{aligned} & |\langle u | (1-v)\delta \rangle| + |\langle u | w(1-v)\delta \rangle| \\ & \leq \epsilon(2 + \epsilon) + \epsilon(1 + \epsilon)(2 + \epsilon) \leq 9\epsilon \end{aligned}$$

since $\text{supp}((1-v)\delta) \cap K = \text{supp}(w(1-v)\delta) \cap K = \emptyset$. Hence we have $\|u - wu\| < 9\epsilon$. Since $A(G)$ is a closed ideal in $B(G)$ it follows that $u \in A(G)$ and the proof is complete.

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