CERTAIN NUMERICAL RADIUS CONTRACTION OPERATORS

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ABSTRACT. In this paper an operator T means a bounded linear operator on a complex Hilbert space H . The numerical radius norm $w(T)$ of an operator T , is defined by $w(T) = \sup \{ |\langle Tx, x \rangle| \}$ for every unit vector x in H . An operator T is said to be a numerical radius contraction if $w(T) \leq 1$. We shall give some theorems on certain numerical radius contraction operators and related results in consequence of these theorems.

Our central result is that an idempotent numerical radius contraction is a projection. Finally we prove that a periodic numerical radius contraction is the direct sum of zero and a unitary operator, that is to say, normal and partial isometric.

We shall begin with the following lemma and its consequence.

LEMMA. *If T is idempotent and $w(T) \leq 1$, then T is a projection.*

We shall give two proofs of the Lemma as follows.

FIRST PROOF OF THE LEMMA. Let $y = Tx$ where $x \in R(T)^\perp$. Then for any positive number t we have $T(x + ty) = (1 + t)y$ so that

$$\begin{aligned} t(1 + t)\|y\|^2 &= \langle (1 + t)y, x + ty \rangle = \langle T(x + ty), x + ty \rangle \\ &\leq \|x + ty\|^2 = \|x\|^2 + t^2\|y\|^2. \end{aligned}$$

Hence $t\|y\|^2 \leq \|x\|^2$ so that $y = 0$. Therefore $T = 0$ on $R(T)^\perp$ and $T = 1$ on $R(T)$ so that T is a projection.

To give the second proof of the Lemma we need the following unpublished paper [8] which is cited with Professor Kato's permission.

THEOREM A. *The following two conditions are equivalent*

- (i) $w(T) \leq 1$,
- (ii) $\|e^{zT}\| \leq e^{|z|}$ for all complex numbers z .

PROOF. Here we cite the proof of [8] for sake of convenience. The condition (ii) is equivalent to $\|e^{t(e^{i\theta}T - I)}\| \leq 1$ for any real θ and $t \geq 0$, which is in turn equivalent to that $e^{i\theta}T - I$ is dissipative for any real θ , which is identical with $w(T) \leq 1$.

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SECOND PROOF OF THE LEMMA. By the idempotency of T and Theorem A, we have

$$\left\| I + zT + \frac{z^2}{2!} T^2 + \cdots \right\| = \|I + (e^z - I)T\| \leq e^{|z|}.$$

Since z is an arbitrary complex number we put $z = t$, where t is a real number. We get the following inequality

$$\|e^{-t}I + (1 - e^{-t})T\| \leq 1.$$

Thus we can conclude $\|T\| \leq 1$ as $t \rightarrow \infty$.

Now it is sufficient to show that $T = T^*T$; this method is shown in [4].

$$\begin{aligned} \|Tx - T^*Tx\|^2 &= \|Tx\|^2 - (Tx, T^*Tx) - (T^*Tx, Tx) + \|T^*Tx\|^2 \\ &= \|Tx\|^2 - (T^2x, Tx) - (Tx, T^2x) + \|T^*Tx\|^2 \\ &= \|Tx\|^2 - \|Tx\|^2 - \|Tx\|^2 + \|T^*Tx\|^2 \\ &= \|T^*Tx\|^2 - \|Tx\|^2 \leq 0; \end{aligned}$$

therefore T is a projection. In fact it is well known that an idempotent contraction is a projection.

We shall give a generalization, namely, we can weaken the idempotency of the operator in the Lemma as follows.

THEOREM. *If $T^k = T$ for some integer $k \geq 2$ and if $w(T) \leq 1$, then T is the direct sum of 0 and a unitary operator.*

PROOF. $T^{2(k-1)} = T^{k-2}T^k = T^{k-2}T = T^{k-1}$. This implies that T^{k-1} is an idempotent operator; also $w(T^{k-1}) \leq (w(T))^{k-1} \leq 1$ by power inequality [1], [5], [9]. Therefore $T^{k-1} = P$ is a projection by the lemma. Hence we can decompose $T = T_1 \oplus T_2$ where $T_1 = T|_{R(P)}$ and $T_2 = T|_{N(P)}$. Then we have

$$T = T^{k-1}T = (T_1^{k-1} \oplus T_2^{k-1})(T_1 \oplus T_2) = (I \oplus 0)(T_1 \oplus T_2) = T_1 \oplus 0.$$

Now $T_1^{k-1} = I$; therefore $w(T_1^{-1}) = w(T_1^{k-2}) \leq (w(T_1))^{k-2} \leq 1$, so that T_1 and T_1^{-1} have numerical radius ≤ 1 ; hence T_1 is unitary [10], so the proof is complete.

COROLLARY 1. *If $T^k = T$ and $w(T) \leq 1$, then T^{k-1} is a projection.*

This proof is contained in the proof of the Theorem.

We remark that Corollary 1 shows that if T is an idempotent operator that satisfies any of the following conditions:

- (i) T is a contraction,
 - (ii) T has equal norm and spectral radius (normaloid [5]),
 - (iii) T has equal numerical and spectral radius (spectraloid [5]),
- then T is a projection.

Moreover we remark that Theorem shows that if T is a periodic operator that satisfies any of the above conditions (i), (ii) and (iii), then T is normal and partial isometric.

COROLLARY 2 [4]. If T is paranormal (i.e., if $\|T^2x\| \geq \|Tx\|^2$ for all unit vectors x ; see [2], [3], [6], [7]) and if $T^k = T$ for some integer $k \geq 2$, then T is normal and partial isometric.

PROOF. If T is paranormal, then T has norm equal to its spectral radius [2], [3], [7]. Hence if $T^k = T$ for some $k \geq 2$, then T is normal and partial isometric by the above remark.

It is known that this class of paranormal operators properly includes that of hyponormal operators ($T^*T \geq TT^*$) and is properly included in the class of normaloids [2].

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ADDED IN PROOF. A generalization of the results in this paper will appear in the following: T. Furuta, *Some theorems on unitary p -dilations of Sz.-Nagy and Foiaş* (to appear in Acta Sci. Math.).

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