SOME APPLICATIONS OF THE TURÁN- KUBILIUS INEQUALITY

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ABSTRACT. The Turán-Kubilius inequality is used to obtain estimates for sums of reciprocals and quotients of prime independent additive arithmetical functions.

Let $g(n)$ be a real valued arithmetical function for which $g(0)=0$ and $g(1) \neq 0$. Define $f(1)=0$ and $f(n)=g(a_1)+\cdots+g(a_r)$ for $n>1$, where $n=p_1^{a_1}\cdots p_r^{a_r}$ is the representation of n as a product of powers of distinct primes. Then $f(n)$ is a prime independent additive arithmetical function and all such functions are of this form. Such functions have been considered previously [1]–[8] and are generalizations of the well-known functions $\omega(n)=r$ and $\Omega(n)=a_1+\cdots+a_r$ [9]. In particular, if $g(n)=O(2^{n/2})$, then [5], [7]

$$(1) \quad \sum_{n \leq x} f(n) = g(1)x \log \log x + Ax + O(x/\log x).$$

If $h(n)$ is an arbitrary additive arithmetical function, then the Turán-Kubilius inequality [10] states that

$$(2) \quad \sum_{m \leq n} |h(m) - A_n|^2 \leq B_n B_n,$$

where

$$A_n = \sum_{p^a \leq n} h(p^a)/p^a \quad \text{and} \quad B_n = \sum_{p^a \leq n} |h(p^a)|^2/p^a.$$

The purpose of this note is to derive a more explicit form of (2) for $f(n)$ and apply this result to the study of sums of reciprocals and quotients of such functions.

THEOREM 1. *If $g(n)=O(b^{n/2})$, $0 < b < 2$, then*

$$(3) \quad \sum_{n \leq x} [f(n) - g(1) \log \log x]^2 \leq Cx \log \log x.$$

PROOF. If $g(n)=O(b^{n/2})$, then

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$$\begin{aligned}
 B_n &= \sum_{p^a \leq n} f^2(p^a)/p^a = \sum_{p^a \leq n} g^2(a)/p^a \\
 &= g^2(1) \sum_{p \leq n} 1/p + \sum_{p^a \leq n}^* g^2(a)/p^a \\
 &= g^2(1) \log \log n + O(1),
 \end{aligned}$$

where $\sum^*$ denotes a sum over $a \geq 2$, since

$$\begin{aligned}
 \sum_{p^a \leq n}^* g^2(a)/p^a &= O \left[\sum_{p^a \leq n}^* (b/p)^a \right] = O \left[\sum_p \sum_{a=2}^{\infty} (b/p)^a \right] \\
 &= O \left[\sum_p 1/p(p-b) \right] = O(1).
 \end{aligned}$$

Similarly, $A_n = g(1) \log \log n + O(1)$. Also,

$$\begin{aligned}
 \sum_{m \leq n} [f(m) - g(1) \log \log n]^2 &= \sum_{m \leq n} [\{f(m) - A_n\} + \{A_n - g(1) \log \log n\}]^2 \\
 &= \sum_{m \leq n} [\{f(m) - A_n\} + O(1)]^2 \\
 &= \sum_{m \leq n} [f(m) - A_n]^2 + O \left[\sum_{m \leq n} f(m) \right] + O(n \log \log n)
 \end{aligned}$$

and (3) follows from (1) and (2).

The following result is proved in the same manner [9] as in the special case $f(n) = \omega(n)$.

COROLLARY. *The normal order of $f(n)$ is $g(1) \log \log n$.*

The following theorems extend and generalize some previous results [6].

THEOREM 2. *If $g(n) = O(b^{n/2})$, $0 < b < 2$, and $g(n) \geq t > 0$, then there exist positive constants c_1 and c_2 such that*

$$(4) \quad c_1 x / \log \log x \leq \sum_{n \leq x} 1/f(n) \leq c_2 x / \log \log x.$$

PROOF.

$$\begin{aligned}
 \sum_{n \leq x} 1/f(n) &= \sum_{n \leq x; 2f(n) < g(1) \log \log x} 1/f(n) + \sum_{n \leq x; 2f(n) \geq g(1) \log \log x} 1/f(n) \\
 &\leq t^{-1} \sum_{n \leq x; 2f(n) < g(1) \log \log x} 1 + 2x/g(1) \log \log x \\
 &\leq (4Ct^{-1} + 2g(1))x/g^2(1) \log \log x,
 \end{aligned}$$

since, by (3),

$$\begin{aligned} Cx \log \log x &\geq \sum_{n \leq x; 2f(n) < g(1) \log \log x} [f(n) - g(1) \log \log x]^2 \\ &\geq (1/4)g^2(1)(\log \log x)^2 \sum_{n \leq x; 2f(n) < g(1) \log \log x} 1 \end{aligned}$$

and $f(n) \geq t > 0$ for $n > 1$ if and only if $g(n) \geq t > 0$ for $n \geq 1$. Also,

$$\sum_{n \leq x} 1/f(n) \geq [2g(1) \log \log x]^{-1} \sum_{n \leq x; f(n) < 2g(1) \log \log x} 1.$$

But

$$\sum_{n \leq x; f(n) < 2g(1) \log \log x} 1 = \sum_{n \leq x} 1 - \sum_{n \leq x; f(n) \geq 2g(1) \log \log x} 1 = x + o(x)$$

by the above corollary and the desired result follows.

THEOREM 3. *If $g(n) = O(n^k)$, then*

$$(5) \quad \sum_{n \leq x} f(n)/\omega(n) = g(1)x + O(x/\log \log x).$$

PROOF.

$$\begin{aligned} \sum_{n \leq x} f(n)/\omega(n) &= \sum_{n \leq x} \sum_{p^m | n} \{g(m) - g(m-1)\}/\omega(n) \\ &= g(1) \sum_{n \leq x} \sum_{p|n} 1/\omega(n) + O \left[\sum_{n \leq x} \sum_{p^m | n}^* m^k/\omega(n) \right] \\ &= g(1) \sum_{n \leq x} 1 + O \left[\sum_{p^m \leq x}^* \sum_{n \leq xp^m} m^k/\omega(np^m) \right] \\ &= g(1)x + O \left[\sum_{p^m \leq x}^* \sum_{n \leq xp^m} m^k/\omega(n) \right] \\ &= g(1)x + O \left[\sum_{n \leq x} \sum_{p^m \leq x/n}^* m^k/\omega(n) \right] \end{aligned}$$

where $\sum^*$ denotes a sum over $m \geq 2$. Now

$$\begin{aligned} \sum_{p^m \leq t}^* m^k &= O \left[\sum_{p \leq t^{1/2}} \sum_{m \leq \log t / \log p} m^k \right] \\ &= O \left[\sum_{p \leq t^{1/2}} (\log t / \log p)^{k+1} \right] = O(t^{1/2}). \end{aligned}$$

It follows by partial summation that [9, Theorem 421]

$$\sum_{n \leq x} f(n)/\omega(n) = g(1)x + O\left[\sum_{n \leq x} 1/\omega(n) + x^{1/2} \int_1^x t^{-3/2} \sum_{n \leq t} 1/\omega(n) dt\right]$$

and (4) implies (5).

It should be observed that (5) is applicable to some functions considered previously [1]–[4]. It is also of interest to note that a new proof of the Turán-Kubilius inequality, based on the inequality of the large sieve, has been given recently by Elliott [11].

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