A SHORT PROOF OF THE F. AND M. RIESZ THEOREM

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In this note we provide a short proof of the following theorem due to F. and M. Riesz:

THEOREM. *Let μ be a complex Borel measure on the unit circle T in the complex plane such that $\int p d\mu = 0$ for all polynomials p . Then μ is absolutely continuous with respect to Lebesgue measure on T .*

PROOF. Let E be a Borel set on T with length zero and let F be a compact subset of E . It suffices to prove that $\mu(F) = 0$. Choose an integer n and cover F by open discs $\Delta_1, \dots, \Delta_N$ with centers $z_1, \dots, z_N$ in F and radii $r_1, \dots, r_N$ respectively such that $\sum_{i=1}^N r_i < 1/n^2$. Put $\rho_i = n \cdot r_i$ and define

$$g_n(z) = 1 - \prod_{i=1}^N \frac{z - z_i}{z - (1 + \rho_i)z_i}.$$

Then $|g_n(z)| \leq 2$ for $z \in T$ and g_n can be uniformly approximated on T by polynomials, so $\int g_n d\mu = 0$. If $z \in F$ then $z \in \Delta_i$ for some i , and hence $|g_n(z) - 1| \leq r_i/\rho_i = 1/n$. If $z \in T$ and $|z - y| \geq \delta > 0$ for all $y \in F$, then we have

$$\begin{aligned} |g_n(z)| &= \left| 1 - \prod_{i=1}^N \left(1 + \frac{\rho_i z_i}{z - (1 + \rho_i)z_i} \right) \right| \\ &\leq \prod_{i=1}^N \left(1 + \frac{\rho_i}{|z - (1 + \rho_i)z_i|} \right) - 1 \\ &\leq \exp \left(\sum_{i=1}^N \log(1 + \rho_i/\delta) \right) - 1 \leq \exp \left(\sum_{i=1}^N \rho_i/\delta \right) - 1 \\ &= \exp \left(n\delta^{-1} \sum_{i=1}^N r_i \right) - 1 < \exp \left(\frac{1}{n\delta} \right) - 1. \end{aligned}$$

So $g_n(z) \rightarrow 1$ for $z \in F$ and $g_n(z) \rightarrow 0$ for $z \in T \setminus F$ as $n \rightarrow \infty$. By dominated convergence, $\mu(F) = \lim_{n \rightarrow \infty} \int g_n d\mu = 0$, and the proof is complete.

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