SHORTER NOTES

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A NOTE ON THE BLUM-HANSON THEOREM

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ABSTRACT. Let T be a linear contraction on an arbitrary Hilbert space. We prove that the weak convergence of the sequence $\{T^n x\}$ implies the strong convergence of the averages of the sequences $\{T^{k_i} x\}$ for all strictly increasing sequences $\{k_i\}$ of positive integers.

In [1] Blum and Hanson proved the following result for a measure-preserving transformation τ on a finite measure space $(X, \mathfrak{B}, m)$ and the induced unitary operator U on $L_2(X, \mathfrak{B}, m)$ defined by $Uf(x) = f(\tau x)$: τ is mixing if and only if $(1/N) \sum_{i=1}^N U^{k_i} f \rightarrow \int_X f dm$ in the L_2 norm for all strictly increasing sequences $\{k_i\}_{i=1}^\infty$ and all f in $L_2(X, \mathfrak{B}, m)$.

We extend here this result to all linear contractions of a Hilbert space and give a short proof. For a generalization in another direction see [2].

THEOREM. *Let T be a linear operator on a Hilbert space H with $\|T\| \leq 1$. Then for any x in H , $T^n x \rightarrow x_0$ weakly if and only if $(1/N) \sum_{i=1}^N T^{k_i} x \rightarrow x_0$ strongly for all strictly increasing sequences $\{k_i\}_{i=1}^\infty$ of positive integers.*

PROOF. If $T^n x \rightarrow x_0$ weakly, then there is y in H and $\epsilon > 0$ and a sequence of integers $k_1 < k_2 < \dots$ such that $\operatorname{Re}(T^{k_i} x - x_0, y) > \epsilon$ for each i . But then

$$\left\| \frac{1}{N} \sum_{i=1}^N T^{k_i} x - x_0 \right\| \geq \operatorname{Re} \left(\frac{1}{N} \sum_{i=1}^N T^{k_i} x - x_0, \frac{y}{\|y\|} \right) > \frac{\epsilon}{\|y\|}.$$

Conversely, suppose that $T^n x \rightarrow x_0$ weakly. Since x_0 is a fixed point of T we can assume $x_0 = 0$. Since T is a contraction $\inf \|T^n x\| = \lim \|T^n x\| = \alpha$. If $\alpha = 0$ we are done, so (by linearity of T) we assume $\alpha = 1$. Given $\epsilon > 0$ there are two positive integers M and K such that $\|T^M x\|^2 < 1 + \epsilon$ and $|(T^M x, T^{M+k} x)| < \epsilon$ for $k \geq K$. Then for $m \geq M$ and $n \geq K + m$ we have

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$$\begin{aligned}
& 2 \operatorname{Re}(T^m x, T^n x) \\
&= \|T^m x + T^n x\|^2 - \|T^m x\|^2 - \|T^n x\|^2 \\
&\leq \|T^{m-M}(T^M x + T^{n-m+M} x)\|^2 - 2 \leq \|T^M x + T^{n-m+M} x\|^2 - 2 \\
&\leq \|T^M x\|^2 + \|T^{n-m+M} x\|^2 + 2 \operatorname{Re}(T^M x, T^{n-m+M} x) - 2 \leq 4\epsilon.
\end{aligned}$$

If $\{k_i\}_{i=1}^\infty$ is any strictly increasing sequence, then we have

$$\begin{aligned}
\left\| \frac{1}{N} \sum_{i=1}^N T^{k_i} x \right\|^2 &= \operatorname{Re} \left(\frac{1}{N} \sum_{i=1}^N T^{k_i} x, \frac{1}{N} \sum_{j=1}^N T^{k_j} x \right) = \frac{1}{N^2} \sum_{i,j} \operatorname{Re}(T^{k_i} x, T^{k_j} x) \\
&= \frac{1}{N^2} \sum_{i=1}^N (T^{k_i} x, T^{k_i} x) + \frac{2}{N^2} \sum_{i < j} \operatorname{Re}(T^{k_i} x, T^{k_j} x) \\
&= \frac{1}{N^2} \sum_{i=1}^N \|T^{k_i} x\|^2 + \frac{2}{N^2} \sum_{i < M; j > i} \operatorname{Re}(T^{k_i} x, T^{k_j} x) \\
&\quad + \frac{2}{N^2} \sum_{i \geq M; i < j < K+i} \operatorname{Re}(T^{k_i} x, T^{k_j} x) \\
&\quad + \frac{1}{N^2} \sum_{i \geq M; j \geq K+i} 2 \operatorname{Re}(T^{k_i} x, T^{k_j} x) \\
&\leq \frac{\|x\|^2}{N} + \frac{2(M-1)N\|x\|^2}{N^2} + \frac{2(K-1)N\|x\|^2}{N^2} + 4\epsilon,
\end{aligned}$$

since $k_i \geq i$ and $k_j - k_i \geq j - i$. Now, for a sufficiently large N the sum of the above terms is small, hence the Theorem.

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