GEOMETRICAL REALIZATION OF COMBINATORIAL GEOMETRIES

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ABSTRACT. A method of depicting combinatorial geometries geometrically is used to construct examples of small combinatorial geometries which arise as subsets of a vector space over division rings restricted by their characteristic.

A *combinatorial pregeometry* $(E, \mathcal{C})$ consists of a finite set E and an identified collection $\mathcal{C}$ of subsets called *circuits* satisfying the property that if C_1 and C_2 are distinct members of $\mathcal{C}$, then C_1 is not contained in C_2 , and if e is a member of $C_1 \cap C_2$, then there exists a circuit C contained in $C_1 \cup C_2 - \{e\}$. A subset of E is called independent if it contains no circuit. It is well known (Whitney [7], Crapo-Rota [1]) that all maximal independent subsets of E have the same cardinality, called the rank of E , and that an integer valued rank function is induced on the subsets of E satisfying:

- (i) $A \subseteq B$ implies $\text{rank } A \leq \text{rank } B$.
- (ii) $\text{rank } A + \text{rank } B \geq \text{rank } A \cup B + \text{rank } A \cap B$.
- (iii) $\text{rank } \emptyset = 0$.
- (iv) $\text{rank } A \cup \{e\} \leq \text{rank } A + 1$.

A *combinatorial geometry* (more briefly *geometry*) is a combinatorial pregeometry in which all circuits have cardinality at least three. As an example the subsets of a fixed subset E of a vector space, which are minimal with respect to being linearly dependent, form the circuits of a pregeometry on E . A geometry which arises in this way is called *representable*, and a fundamental problem is to classify these geometries which are representable over some division ring.

The observation has been made many times (Mac Lane [5], Ingleton [3]) that a projective plane induces a geometry on its points by taking as circuits all sets of three collinear points, and all sets of four coplanar points, no three of which are collinear. It led Ingleton to conjecture that the smallest geometry (in cardinality) which cannot be represented (over any division ring) must have ten points, namely a Desarguesian configuration with one line missing.

It is frequently useful to depict portions of projective planes by

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drawing points in the plane and drawing curves through the sets of points which are collinear. One purpose of this note is to generalize this idea by showing that any combinatorial geometry can be depicted as a finite set of points lying on certain surfaces in Euclidean space. The Euclidean dimension of the surfaces helps to distinguish the rank of the set of points lying on that surface.

The second purpose of this note is to exhibit a number of interesting geometries, among them a class of Lazarsen rank three geometries (Example 3) which are geometries whose representations require division rings with restricted characteristic [4], and a counterexample to Ingleton's conjecture (Example 1). Other counterexamples have been obtained by Vámos [6] and Crapo [2].

Theorem 1 provides a messy looking condition on a collection of families of subsets of a set E , which induces a geometry on the set E . The motivation for the condition is that it enables us to depict the combinatorial geometry geometrically. This is brought out in the useful special cases of rank 3 and rank 4 geometries.

Theorem 2 is the converse of Theorem 1, showing that every geometry is induced by a family of sets satisfying Theorem 1.

THEOREM 1. *Let E be a finite set and for each t , $3 \leq t \leq n+2$, let $P(t)$ denote a family of subsets of E satisfying the following condition.*

() If $A_i \in P(t_i)$ and $A_1 \cap A_2 \neq \emptyset$ then $A_1 \cup A_2 \subseteq A \in P(\lambda)$ where λ is given by*

$$\lambda = t_1 + t_2 - |A_1 \cap A_2| - 1 \text{ if } \{r: |A_1 \cap A_2| \geq r, A_1 \cap A_2 \subseteq A \in P(r)\} = \emptyset, \\ = t_1 + t_2 - \min\{r: |A_1 \cap A_2| \geq r, A_1 \cap A_2 \subseteq A \in P(r)\} \text{ otherwise}$$

and $P(n+2) = E$.

Then $(E, \mathcal{C})$ is a geometry with rank $E \leq n+1$ where $\mathcal{C}$ consists of the minimal members (under inclusion) of $\{C: C \subseteq A \in P(t) \text{ and } |C| \geq t\}$.

The members of $P(t)$ turn out to be dependent subsets of $(E, \mathcal{C})$ maximal with respect to having rank $t-1$, and the formula for λ , which states how these sets fit together is just a restatement of the submodular property of the rank function.

PROOF. Let $C_i \subseteq A_i \in P(t_i)$ be members of $\mathcal{C}$. Then $|C_i| = t_i$ and if $e \in C_1 \cap C_2$, $C_1 \neq C_2$, then to verify the circuit axioms, we need only show that $D = C_1 \cup C_2 - \{e\}$ contains a member of $\mathcal{C}$. Now we are given that $D \subseteq A \in S(\lambda)$ where λ is given by the formula. If $\lambda = t_1 + t_2 - r$, then $|C_1 \cap C_2| \leq r - 1$ since $C_1 \cap C_2$ does not contain a member of $\mathcal{C}$ by minimality. Then since $|D| = t_1 + t_2 - 1 - |C_1 \cap C_2|$ we get $|D| \geq t_1 + t_2 - r = \lambda$. And if $\lambda = t_1 + t_2 - |A_1 \cap A_2| - 1$ then $|C_1 \cap C_2| \leq |A_1 \cap A_2|$ whence $|D| \geq \lambda$. Thus in either case D contains a member

of $\mathcal{C}$. Since $P(n+2) = E$, every subset of E with cardinality $n+2$ is dependent, and so a basis of $(E, \mathcal{C})$ must have cardinality $\leq n+1$.

THEOREM 2. *Let $(E, \mathcal{C})$ be a combinatorial geometry and denote by $P(t)$ the family of maximal members of $\{A \subseteq E: \text{rk } A = t-1 < |A|\}$. Then the collection of families $P(t)$, $3 \leq t \leq \text{rk } E + 1$, satisfy the condition (*).*

PROOF. Since $(E, \mathcal{C})$ is a geometry, $P(1) = P(2) = \emptyset$. We are given that $\text{rank } A_i = t_i - 1$, $|A_i| \geq t_i$ and $A_1 \cap A_2 \neq \emptyset$. Put $W(A_1 \cap A_2) = \{r: |A_1 \cap A_2| \geq r, A_1 \cap A_2 \subseteq A \in P(r)\}$. We need to verify that

(i) if $W(A_1 \cap A_2) = \emptyset$ then $\text{rank } A_1 \cup A_2 = t_1 + t_2 - |A_1 \cap A_2| - 2 < |A_1 \cup A_2|$ and

(ii) if $W(A_1 \cap A_2) \neq \emptyset$ then $\text{rank } A_1 \cup A_2 = t_1 + t_2 - \min W(A_1 \cap A_2) - 1 < |A_1 \cup A_2|$.

To establish (ii), let $t = \min W(A_1 \cap A_2)$ and note that $\text{rank } A_1 \cap A_2 = t-1$. Since $|A_i| \geq |A_1 \cap A_2| + t_i - t$ we find that

$$|A_1 \cup A_2| \geq t_1 + t_2 - 2t + |A_1 \cap A_2| \geq t_1 + t_2 - t.$$

Now let X be a maximal independent subset of $A_1 \cup A_2$ containing a maximal independent subset of $A_1 \cap A_2$. The maximality of the A_i in the definition of $P(t_i)$ assures us that

$$|X| \geq t_1 - 1 + t_2 - 1 - (t - 1) = t_1 + t_2 - t - 1.$$

Suppose by way of contradiction, that $|X| \geq t_1 + t_2 - t$. Then we would have

$$\begin{aligned} |X \cap A_1 \cap A_2| &= |X \cap A_1| + |X \cap A_2| - |X \cap (A_1 \cup A_2)| \\ &\leq t_1 - 1 + t_2 - 1 - (t_1 + t_2 - t) \leq t - 2. \end{aligned}$$

Thus $\text{rank } A_1 \cap A_2 \leq t-2 \leq |A_1 \cap A_2|$ contradicting the minimality of t .

Therefore $\text{rank } A_1 \cup A_2 = |X| = t_1 + t_2 - t - 1$ and (ii) is established.

To verify (i), a similar argument shows that

$$\text{rk } A_1 \cup A_2 \geq t_1 + t_2 - |A_1 \cap A_2| - 2$$

and

$$\begin{aligned} \text{rank } A_1 \cup A_2 &\leq \text{rank } A_1 + \text{rank } A_2 - \text{rank } A_1 \cap A_2 \\ &\leq t_1 - 1 + t_2 - 1 - |A_1 \cap A_2| < |A_1 \cup A_2|. \end{aligned}$$

To check that the collection $P(t)$ does induce the original geometry $(E, \mathcal{C})$ we notice that X contains a circuit induced from $P(t)$ if and only if $|X| > \text{rk } X$, i.e. if and only if X contains a member of $\mathcal{C}$.

Geometric realization. Suppose that for $3 \leq t \leq n+2$, $S(t)$ is a family of surfaces with Euclidean dimension $t-2$. Further, let E be a finite set of points and suppose the collection $P(t) = E \cap S(t)$, $3 \leq t \leq n+2$, satisfy condition (*). Then by virtue of Theorem 1, we can say that the collection of surfaces $S(t)$, $3 \leq t \leq n+2$, induce a geometry on E . The content of Theorem 2 is that every geometry can be so realized since the only restrictions on the surfaces $S(t)$ are their pairwise intersections in the finite set E . The following two special cases illustrate the situation for rank 3 and rank 4 geometries.

Special case: rank 3. Let E be a finite set of points in the Euclidean plane, $S(3)$ a set of curves in the plane such that any two distinct curves meet in at most one point of E , and $S(4)$ the plane. Then the induced geometry $(E, \mathcal{C})$ has rank ≤ 3 .

PROOF. Calculation based on the given intersections shows that, for $3 \leq t_1 \leq t_2 \leq 4$, $\lambda = 4$, i.e. that the union of any two distinct intersecting "surfaces" must lie in $S(4)$, which they do by definition.

Special case: rank 4. Let E be a finite set of points in Euclidean 3-space. Let $S(5)$ be the whole 3-space, $S(4)$ a set of surfaces and $S(3)$ a set of curves satisfying:

- (i) any two distinct surfaces which meet in more than two points of E do so in a curve in $S(3)$;
- (ii) any distinct curves which intersect in a point of E do so in at most one and lie on a common surface;
- (iii) any curve not lying on a surface intersects it in at most one point of E .

Then the induced space has rank ≤ 4 .

PROOF. Using the intersection information, one calculates the table of values and checks that the two surfaces represented by t_1 and t_2 do lie on the surface represented by λ .

t_1	t_2	λ
3	3	4
3	4	4, 5
4	4	5

EXAMPLE 1. (Counterexample to Ingleton's conjecture.) Let $E = \{A, B, C, D, A', B', C', D'\}$ where A, B, C, D are the vertices of a regular tetrahedron in 3-space, and A', B', C', D' are the centroids of the opposite faces, respectively. Put $S(5) = 3\text{-space}$, $S(4) = \text{planes of } 3\text{-space except the plane through } ABA'B'$, and $S(3) = \emptyset$. The

induced rank 4 cardinality 8 geometry $(E, \mathcal{C})$ cannot be represented over any division ring.

PROOF. Theorem 1 guarantees that we have a geometry. Assume that it is representable over a division ring R in a left vector space. Without loss of generality we may select a basis of the vector space so that $A = (1, 0, 0, 0)$, $B = (0, 1, 0, 0)$, $C = (0, 0, 1, 0)$, $D = (0, 0, 0, 1)$ and $C' = (1, 1, 0, 1)$. Since $\{A, B, C, D'\}$ and $\{D, C, D', C'\}$ are circuits, we can choose $D' = (1, 1, x, 0)$ where $0 \neq x \in R$. Then the circuits $\{A', B, C, D\}$, $\{A, C, A', C'\}$ and $\{A, D, A', D'\}$ force $A' = (0, 1, x, 1)$ and similarly $B' = (1, 0, x, 1)$. But then $\{A, B, A', B'\}$ would be linearly dependent contradicting our construction. Choosing $S(4)' = S(4) \cup \{\text{inscribed sphere}\}$, a distinct rank 4 geometry is induced which also cannot be represented.

REMARK. The fact that any nonrepresentable geometry must have at least eight points can be verified using the enumeration of all eight point and smaller geometries [2], or by a more tedious enumeration based on Theorem 2.

EXAMPLE 2. Let $E = \{A, B, C, A', B', C', X, Y, Z\}$ be nine points in the plane as shown in Figure 1. Put $S(4) = \text{plane}$.

$S(3) = \text{solid lines, inducing } (E, \mathcal{C})$,

$S(3)^* = S(3)$ with a circle through C, C', X inducing $(E, \mathcal{C}^*)$,

$S(3)^{**} = S(3)^*$ with a circle through A, A', Z , inducing $(E, \mathcal{C}^{**})$,

$S(3)^{***} = S(3)^*$ with the line through X, Y, Z , inducing $(E, \mathcal{C}^{***})$.

FIGURE 1

Then none of the four distinct rank 3 cardinality 9 geometries can be represented over any division ring.

PROOF. Assume they were representable. Without loss of generality A', B, C' and Y can be chosen as shown. Let $X = (1, x, 0)$ for some x in the division ring. Then one gets $A = (x, x, x-1)$, $C = (x, x-1, x-1)$

and $Z = (0, x-1, -1)$ whence X, Y, Z would be linearly dependent contradicting the construction of the first three geometries. Also, $\{C, C', X\}$ is a circuit if and only if $x^2 - x + 1 = 0$, and A, A', Z is a circuit if and only if $x^2 - x + 1 = 0$. Thus $(E, \mathcal{C}^{***})$ is not representable either.

EXAMPLE 3. (Lazarson geometries.) Let $E_n = \{M, P, Q, R, X, Y, Z\} \cup \{A_i, B_i : 1 \leq i \leq n\}$ be $7+2n$ points in the plane as shown in Figure 2, with $S(4) = \text{plane}$, $S(3) = \text{lines shown}$. Without loss of generality, in any representation we may choose X, Y, Z, M and hence P, Q, R as shown. One easily deduces that $A_i = (1, i+1, i)$ and $B_i = (1, i, i+1)$. By adjoining to $S(3)$ a curve through A_i, B_i, Z we induce a geometry on E_n which requires the characteristic of the division ring to divide

FIGURE 2

$2i+1$. By taking two such curves, corresponding to i and j with $2i+1, 2j+1$ distinct primes, the induced geometry becomes non-representable. Thus we have geometries with cardinality $6+p$, rank 3 requiring characteristic p (for odd primes p)—improving Lazarson's examples with cardinality $2p+3$, rank $p+1$ requiring characteristic p .

An outstanding problem is to find a geometry which is representable, but only over two distinct characteristics. Example 3 does not succeed because if you choose $2i+1$ to be divisible by two primes p and q , then the geometry is not representable over the larger prime.

(If it were, we would have the point $A_p = (1, p+1, p) = (1, 1, 0) = Q$ which is not the case.)

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