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THE DIFFERENTIAL IDEALS $[y^p z]$

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ABSTRACT. In this paper we obtain Levi bases for $[y^2 z]$ and $[y^3 z]$, and develop the associated critical weight functions.

1. **Introduction.** In the appendix of his book [11], Ritt gives a a list of topics to be investigated, and the first of these indicates the need to study some special differential ideals. An apparently reasonable first goal in such an investigation is to study principal differential ideals $[M]$ generated by a monomial M of zero weight. This work was started by Levi [2], who studied the ideals $[y^p]$, $p=2, 3, \cdots$, and $[uv]$, where y, u and v are differential indeterminates. For each of these ideals he defines a critical weight as a function of the degree of a term, and shows that a term with weight less than the critical weight is in the ideal and that for every weight greater than or equal to the critical weight there is a term not in the ideal. Further light is shed on these ideals in several papers by Mead [3], [4], [5] and by the O'Keefe's [6], [7], [8], [9], [10]. An extension of Levi's results to the ideals $[u, v \cdots z]$ generated by a product of n differential indeterminates is in a paper by Hillman, Mead, K. B. O'Keefe and E. S. O'Keefe [1]. All of the work mentioned so far applies to rings of differential polynomials over any field F of characteristic zero, i.e., any field containing the rational numbers Q.

In this paper we deal with the differential ring $Q\{y, z\}$, but the results apply to $F\{y, z\}$, where F is any field that contains Q. We consider the ideals $[y^p z]$, $p=2, 3, 4, \cdots$, but obtain results comparable to Levi's only in the cases $p=2$ and $p=3$.

The notation in this paper is that of Levi [2]. Also, the structures of the proofs in this paper are essentially the work of Levi [2]. It was only necessary to indicate the basic syzygies and to define orders, β-term and γ-term for the ideals $[y^2 z]$ and $[y^3 z]$ so as to fit Levi's structures.

2. **Some basic definitions.** Let $A = y^p z$ for some fixed p. Then the kth derivative of A is

$$A_k = \sum c(i_1, i_2, \cdots, i_p, j) y_{i_1} y_{i_2} \cdots y_{i_p} z_j$$

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where the subscripts on y and z indicate differentiation and where the sum is taken over all choices of $i_1, \cdots, i_p, j$ such that $0 \leqq i_1 \leqq i_2 \leqq \cdots \leqq i_p$, $0 \leqq j$ and $j + \sum_{n=1}^{p} i_n = k$. All the coefficients $c(i_1, \cdots, i_p, j)$ are integers which can be found easily by Liebniz' rule.

In writing a p.p. (i.e., a power product)

$$E = y_0^{a_0} y_1^{a_1} \cdots y_m^{a_m} z_{j_1} z_{j_2} \cdots z_{j_e} \tag{1}$$

in y, z and their derivatives, it will always be assumed that $j_1 \leqq j_2 \leqq \cdots \leqq j_e$. Let $d = \sum_{i=0}^{m} a_i$. Then (d, e) is called the *signature* of E, and $w = \sum_{i=1}^{m} i a_i + \sum_{n=1}^{e} j_n$ is called the *weight* of E.

We now define a *β-factor* of E with respect to the ideal $[y^p z]$:

(i) If $i < e$ and $a_i \geqq p$, then $y_i^p z_{j_{i+1}}$ is a β-factor.

(ii) If $i < e$, $0 < a_{i-1} < p$, $a_{i-1} + a_i \geqq p$ and $j_{i+1} - j_i = r \leqq \min\{p-2, a_{i-1}+a_i-p\}$, then, letting $s = p + r - a_{i-1}$, $y_{i-1}^{a_{i-1}} y_i^s z_{j_{i+1}}$ is a β-factor if $s < p$.

E is said to be a *β-term* if it has one or more β-factors. If E is not a β-term, it will be called an *α-term*.

We next define a partial ordering of the p.p. Let E be the p.p. of (1) above and let $E' = y_0^{a_0'} \cdots y_n^{a_n'} z_{j_1'} \cdots z_{j_e'}$. We compare E and E' by looking at

$$j_1' - j_1, \quad a_0 - a_0', \quad j_2' - j_2, \quad a_1 - a_1', \cdots, j_e' - j_e, \quad a_{e-1} - a_{e-1}'.$$

If the first nonzero difference in this sequence is positive, we say E is lower than E' (or E' is higher than E) and write $E \ll E'$ (or $E' \gg E$). This relation is clearly transitive.

3. **The congruence lemma.** We are now ready to prove

LEMMA 1. *For $p = 2, 3$, any β-term is congruent modulo $[y^p z]$ to a linear combination (with rational coefficients) of α-terms of the same weight and signature.*

The proof is the same as the proof of the corresponding lemma in Levi's paper [2], with one exception. With $p = 3$, a β-factor $y_{i-1}^2 y_i^2 z_{j_{i+1}}$ with $j_i + 1 = j_{i+1}$ seems to offer a choice of substituting for $y_{i-1}^2 y_i z_{j_{i+1}}$ or for $y_{i-1} y_i^2 z_{j_{i+1}}$. It is, in fact, necessary to substitute both ways so as to eliminate a higher term which would arise if either substitution were made alone. This same technique will be needed in the proof of Lemma 2 where there may be several choices of substitution and, in general, it will be necessary to employ all of them.

4. **A partial ordering of** $[y^p z]$. We now return to the general case of $p \geqq 2$. Let

$$H = E\mathfrak{A} = y_0^{a_0} \cdots y_r^{a_r} z_{j_1} \cdots z_{j_s} A_0^{b_0} \cdots A_t^{b_t} \tag{2}$$

with at least one of the $b_k \neq 0$. Note that any such term is in $[y^p z]$. We wish to define a partial ordering on forms of this type, but we must first develop some preliminary definitions. First we let $j_0 = 0$ and $j_{s+1} = \infty$, and now define

$$f(n) = pn - 2 + j_n, \qquad n = 0, 1, \cdots, s + 1.$$

Since $f(n)$ is an increasing function, for a given $k \geqq 0$, there exist unique integers n and m with $0 \leqq n \leqq s$ and $0 \leqq m < p + (j_n - j_{n-1})$, such that $k = f(n) + m$. It is clear that f is a function of E as well as of n; therefore, if the context does not make clear which E is meant, we will use the notation $f(E, n)$. Now define $B_0 = \prod_{k=0}^{f(1)} A_k^{b_k}$ and $B_n = \prod_{k=f(n)+p}^{f(n+1)} A_k^{b_k}$ for $n = 1, 2, \cdots, s$. The H of (2) above can now be expressed as

$$H = EB_0 A_{f(1)+1}^{b_{f(1)+1}} \cdots A_{f(1)+p-1}^{b_{f(1)+p-1}} B_1 A_{f(2)+1}^{b_{f(2)+1}} \cdots B_s.$$

Suppose $B = \prod_{k=M}^{N} A_k^{b_k}$ and $B' = \prod_{k=M}^{N'} A_k^{b_k'}$. We wish to define an order on these forms. Note that for two such forms to be compared they must have the same lower limit M in the product, but may have different upper limits N and N'. We look at the sequence of differences $b_k - b_k'$, $k = M, M+1, \cdots, \min\{N, N'\}$. If the first such nonzero difference is positive, we say B is lower than B' and write $B \ll B'$ (or B' is higher than B and write $B' \gg B$). If the differences are all zero and $N < N'$, then $B \ll B'$. If $N = N'$ and $b_k = b_k'$, $k = M, M+1, \cdots, N$, then $B = B'$. It is important to note that $B = \prod_{k=1}^{2} A_k^{b_k}$ and $B' = \prod_{k=1}^{3} A_k^{b_k'}$ with $b_1 = b_1' = 1$ and $b_2 = b_2' = b_3' = 0$ are not equal. In this case $B \ll B'$ since $b_1 - b_1' = b_2 - b_2' = 0$ and $N = 2 < 3 = N'$.

We are now ready to order forms of the type of the H of (2). Let $H' = E'B_0' A_{f(1)+1}^{b'_{f(1)+1}} \cdots B_{s'}'^{b_{s'}'}$ where $E' = y_0^{a_0'} \cdots y_{r'}^{a_{r'}'} z_{j_1'} \cdots z_{j'_{s'}}$. To compare H and H' we compare B_0 and B_0'. If $B_0 \ll B_0'$, we say $H \ll H'$. If $B_0 = B_0'$ (note this implies $f(E, 1) = f(E', 1)$) we look at the differences $b_{f(1)+m} - b'_{f(1)+m}$, $m = 1, 2, \cdots, p-1$, and then at $a_0 - a_0'$. If the first such nonzero difference is positive, we say $H \ll H'$. If the differences are all zero, compare B_1 and B_1'. If $B_1 \ll B_1'$, then $H \ll H'$. If $B_1 = B_1'$, then $f(E, 2) = f(E', 2)$. Now look at the differences $b_{f(2)+m} - b'_{f(2)+m}$, $m = 1, 2, \cdots, p-1$, and $a_1 - a_1'$. If the first such nonzero difference is positive, $H \ll H'$. If the differences are all zero, proceed in this manner to compare B_i and B_i', then look at $b_{f(i+1)+m}-b'_{f(i+1)+1}$, $m=1, 2, \cdots, p-1$, and then a_i-a_i' for $i=2, 3, \cdots, \min\{s, s'\}$. If the first distinction is of the form $B_i \ll B_i'$ or a positive difference, then $H \ll H'$. Since this relation is clearly transitive, and since every form H of (2) is in $[y^pz]$, this is a partial ordering on $[y^pz]$.

5. **Levi bases for $[y^2z]$ and $[y^3z]$.** Still paralleling the work of Levi [**2**], we now distinguish subsets of $[y^2z]$ and $[y^3z]$ which will be called γ-terms and which will be shown to be bases for $[y^2z]$ and $[y^3z]$ as vector spaces over Q. For $p=2$, a term H of (2) will be called a *γ-term* if all of the following conditions are satisfied:

(a) E is an α-term.
(b) If $b_{f(n)+2}\neq 0$, then $a_{n-1}=0$.
(c) If $b_{f(n)+1}\neq 0$, then $a_{n-1}=0$ and
  (c1) if $a_{n-2}\neq 0$, then $j_{n-1}<j_n$.
(d) $b_{f(n)+1}\leqq 1$.

For $p=3$, H will be called a *γ-term* if all of these conditions are satisfied:

(a) E is an α-term.
(b) If $b_{f(n)+2}\neq 0$, then $a_{n-1}\leqq 1$.
(c) If $b_{f(n)+2}\neq 0$, then $a_{n-1}\leqq 1$ and if $a_{n-1}=1$, then
  (c1) $j_n<j_{n+1}$;
  (c2) if $a_{n-2}=1$, then $j_{n-1}<j_n$;
  (c3) if $a_{n-2}=2$, then $j_{n-1}+1<j_n$.
(d) If $b_{f(n)+1}\neq 0$, then $a_{n-1}=0$ and
  (d1) if $a_{n-2}=2$, then $j_{n-1}<j_n$.
(e) If $b_{f(n)}\neq 0$, then $a_{n-1}=0$ and
  (e1) if $a_{n-2}=1$, then $j_{n-1}<j_n$;
  (e2) if $a_{n-2}=2$, then $j_{n-1}+1<j_n$.
(f) If $a_{n-1}=1$, then $b_{f(n)+2}\leqq 1$.
(g) $b_{f(n)}+b_{f(n)+1}\leqq 1$.

We now have

LEMMA 2. *Every H of* (2) *is equal to a sum, with rational coefficients, of γ-terms of the same weight and signature as H.*

The proof is the same as the proof of the corresponding lemma in Levi's paper [**2**], but we need two basic syzygies. The first comes from the relation

$$py_1zA + yz_1A - yzA_1 = 0 \tag{3}$$

where $A=y^pz$. If (3) is differentiated n times and like terms are collected, we obtain

$$\sum C_p(i, j, k) y_i z_j A_k = 0$$

where the sum is over all nonnegative integers i, j, and k such that $i+j+k-1=n$. Similarly, the relation

$$p[yy_2z_1 - (p+1)y_1^2 z_1 - yy_1z_2]A + (2py_1z_1 + yz_2)yA_1 - y^2z_1A_2 = 0$$

can be differentiated to obtain

$$\sum D_p(i_1, i_2, j, k) y_{i_1} y_{i_2} z_j A_k = 0.$$

The coefficients $C_p(i, j, k)$ and $D_p(i_1, i_2, j, k)$ can be found by Liebniz' rule.

6. **The structure of $[y^2z]$ and $[y^3z]$.** Before giving the main results of this section, we need a result equivalent to what Levi [**2**] called the fundamental lemma. For $p=2, 3$, we have

LEMMA 3. *The number of γ-terms of a given weight and signature is less than or equal to the number of β-terms of the same weight and signature.*

Again, the proof is the same as that of the corresponding lemma in Levi's work [**2**]. The necessary injection is essentially the same as that used by Levi in the $[uv]$ case.

Now, as is shown by Levi [**2**], we have

THEOREM 1. *For $p=2$, 3, no finite sum of α-terms with rational coefficients is in $[y^pz]$.*

Now, by examining the minimum weight for an α-term, it can be shown that for $p=2$, 3 the critical weight function, $w(d, e)$, for $[y^pz]$ is given by the following. Let $q=[d/(p-1)]$ and $r=d-q(p-1)$. Then

$$\begin{aligned} w(d, e) &= (p-1)(q-1) + 2qr && \text{if } e \geqq 2q, \\ &= de - (p-1)e/2 - (p-1)e^2/4 && \text{if } e \text{ is even and } e < 2q, \\ &= de - (p-1)e/2 - (p-1)(e^2+1)/4 && \text{if } e \text{ is odd and } e < 2q. \end{aligned}$$

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