# BAIRE FUNCTIONS AND THEIR RESTRICTIONS
## TO SPECIAL SETS

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ABSTRACT. A function $f$ from a complete and separable metric space $X$ into the real numbers is of Baire class 1 iff for every nonempty perfect subset $H$ of $X$, $f|H$ contains a point where $f|H$ is continuous. This paper examines a similar idea obtained by changing "perfect subset $H$" to "union of a countable collection of perfect subsets" in the preceding characterization of Baire class 1 functions. This new idea is also characterized by using "condensation points" and "totally imperfect sets." Functions of this new type are of Baire class 1. However, the converse is false.

**Introduction.** There are various characterizations for functions of Baire class 1 in different spaces. For example see [5, p. 279], [6] and [7]. In a separable and complete metric space $X$, a function of Baire class 1 can be characterized as a function $f: X \to R$ ($R$ denotes the real numbers) such that if $H$ is a nonempty perfect subset of $X$, then $f|H$ (the restriction of $f$ to the domain $H$) contains a point where $f|H$ is continuous [4, p. 289]. Any function described as of Baire class 1 in this paper will mean a function that satisfies the preceding characterization. This paper examines an alteration of the preceding characterization of Baire class 1 and its relation to Baire functions. The alteration is stated in the following definition.

DEFINITION 1. A function $f$ from a separable and complete metric space $X$ into the real numbers has property P means that if $H$ *is the nonempty union of a countable collection of perfect subsets* of $X$, then $f|H$ contains a point where $f|H$ is continuous.

The first theorem establishes two other characterizations of property P. There exist functions of Baire class 1 for which this new condition (property P) is false; the paper shall conclude with an example of such a function. The uniform limit of a sequence of Baire class 1 functions is a function of Baire class 1 [5]. However, the uniform limit of a sequence of functions, each with property P, does not necessarily have property P. Equivalent characterizations of property P using condensation points and totally imperfect sets are derived (a totally imperfect set is a set that contains no perfect set [4, p. 201]).

THEOREM 1. *Suppose $X$ is a separable and complete metric space. Suppose $f$ is a function from $X$ into $R$. The following statements are equivalent*:

(a) *If $H$ is the nonempty union of a countable collection of perfect subsets of $X$, then $f|H$ contains a point where $f|H$ is continuous.*

(b) *If $H$ is a nonempty subset of $X$ such that every point of $H$ is a condensation point of $H$, then $f|H$ contains a point where $f|H$ is continuous.*

(c) *If $H$ is a nonempty totally imperfect subset of $X$ and every point of $H$ is a condensation point of $H$, then $f|H$ contains a point where $f|H$ is continuous.*

DEFINITION 2. The function $f: X \to R$ is a step-like function means that if $B$ is a nonempty subset of $X$, then there is an open set $U$ containing a point of $B$ such that $f|BU$ is constant.

THEOREM 2. *Suppose $X$ is a separable and complete metric space. Suppose $f$ is a function from $X$ into $R$. The following statements are equivalent*:

(a) *$f$ is of Baire class 1.*

(b) *$f$ is the uniform limit of a sequence of step-like functions.*

(c) *$f$ is the uniform limit of a sequence of functions each with property* P.

LEMMA. *If $F$ is a totally imperfect subset of a complete, separable and perfect metric space $X$ and $E$ is a countable subset of $X$, then $F+E$ is totally imperfect.*

1. PROOF OF LEMMA. Suppose $F+E$ is not totally imperfect. Let $D$ denote a nonempty perfect subset of $F+E$. Therefore $D-E$ is a subset of $F$ and an uncountable Borel subset of $X$. Consequently [1], [3], [4, p. 205], $D-E$ contains a homeomorphic image of the Cantor set. Contradiction. Therefore $F+E$ is totally imperfect.

2. PROOF OF THEOREM 1. Proof that (a) implies (b). Suppose (a). Suppose $H$ is a nonempty subset of $X$ such that every point of $H$ is a condensation point of $H$ and $f|H$ is discontinuous at each of its points. For each positive number $c$ let $H_c$ denote the subset of $X$ such that $x \in H_c$ iff the saltus of $f|H+\{(x, f(x))\}$ at $(x, f(x))$ is greater than or equal to $c$. For each $c$, $H_c$ is closed. There exists a positive number $d$ such that $H_d$ is uncountable and if $c$ is a positive number less than $d$, then $H_c$ is uncountable. For each positive number $c \le d$, there exists an uncountable and perfect set $K_c$ and a nonempty countable set $M_c$ such that $H_c = K_c + M_c$ [4, p. 159]. Let $M$ denote the union of the sets $M_{d/1}, M_{d/2}, M_{d/3}, \cdots$. Let $P_1, P_2, P_3, \cdots$ denote a sequence of all

the points of $M$. $N(P, e)$ will denote the neighborhood of the point $P$ with radius $e$. For each positive integer $m$ and each positive integer $n$, $HN(P_n, \frac{1}{2}m)$ is uncountable. Therefore there is a positive integer $i$ such that $H_{d/i}\mathrm{Cl}(N(P_n, \frac{1}{2}m))$ is uncountable and closed. Therefore there is a nonempty perfect subset of $H_{d/i}\mathrm{Cl}(N(P_n, \frac{1}{2}m))$ $\subset H_{d/i}N(P_n, 1/m)$. Let $L_n(m)$ denote one such set for each $m$ and $n$. For each $n$, let $L_n$ denote the union of $L_n(1)$, $L_n(2)$, $L_n(3)$, $\cdots$ and $\{P_n\}$. $L_n$ is a nonempty perfect set.

Let $G$ denote the countable collection of nonempty perfect subsets of $X$ such that $C \in G$ iff there is a positive integer $n$ such that $C = K_{d/n}$ or $C = L_n$. $G^*$ (the union of the elements of $G$) is the nonempty union of a countable collection of perfect subsets of $X$. Therefore $f|G^*$ contains a point $(y, f(y))$ where $f|G^*$ is continuous. Therefore there is an $n$ such that $y \in H_{d/n}$. Therefore $f|H + \{(y, f(y))\}$ is not continuous at $(y, f(y))$. $H$ is a subset of $G^*$. Therefore $f|G^*$ is not continuous at $(y, f(y))$. Contradiction. Therefore (a) implies (b).

Proof that (b) implies (c). Trivial.

Proof that (c) implies (a). Suppose (c). Suppose there is a nonempty countable collection $G$ of nonempty perfect subsets of $X$ such that $f|G^*$ contains no point where $f|G^*$ is continuous. Let $K$ denote the union of all perfect subsets of $X$. Therefore $K$ is an uncountable, perfect, separable and complete subspace of $X$. Therefore $K$ is the sum of two disjoint, and totally imperfect sets $L$ and $M$ [4, pp. 201–202]. Every point of $L$ is a condensation point of $L$. For suppose $P$ is a point of $L$ that is not a condensation point of $L$. There are open sets $U$ and $V$ each containing $P$ such that $UL$ is countable and $\mathrm{Cl}\ V \subset U$. $(\mathrm{Cl}\ V)K$ is closed and uncountable. Therefore $(\mathrm{Cl}\ V)K$ contains a nonempty perfect set. But $(\mathrm{Cl}\ V)K \subset (UL) + M$. Therefore $(UL) + M$ contains a nonempty perfect subset. Therefore, by the Lemma, $M$ is not totally imperfect. Contradiction. Likewise, every point of $L$ is a condensation point of $M$. Likewise, every point of $M$ is a condensation point of $M$ and of $L$. Similarly, it can be shown that every point of $LG^*$ is a condensation point of $LG^*$ and of $MG^*$ and also that every point of $MG^*$ is a condensation point of $MG^*$ and of $LG^*$. $LG^*$ and $NG^*$ are two disjoint, uncountable and totally imperfect sets such that $G^* = LG^* + MG^*$.

In the following argument, a function and its graph are regarded as one and the same. $K \times R$ is separable since $K$ and $R$ are both separable metric spaces. $f|G^*$ is a subset of $K \times R$. Therefore $f|G^*$ is separable [2, pp. 141–142]. Let $E$ denote a countable subset of $G^*$ such that $f|E$ is a countable and dense subset of $f|G^*$. By the Lemma, $LG^* + E$ is a totally imperfect subset of $X$. Also, every point of $LG^*$

$+E$ is a condensation point of $LG^*$. Therefore $f|(LG^*+E)$ contains a point $P$ where $f|(LG^*+E)$ is continuous. But $f|G^*$ is also continuous at $P$ because $f|E$ is dense in $f|G^*$. Contradiction. Therefore (a). Therefore (c) implies (a).

Therefore (a), (b) and (c) are equivalent.

3. Proof of Theorem 2. Proof that (a) implies (b). Suppose $f$ is of Baire class 1. Suppose the collection $T$ of open sets of $X$ is well-ordered by "precedes". If $H$ is a subcollection of $T$ and $U$ is an open set, then $I[H, U]$ will denote $U$ minus the union of the open sets in $H$ that precede the open set $U$.

Definition 3. For each positive integer $n$, a subcollection $H$ of $T$ is of type $A(n)$ means that $H$ contains the first element $U$ of $T$ that has the property that if $x$ and $y$ are in $U$, then $|f(x)-f(y)|<1/n$ and if $V$ is in $H$, then $V$ is the first element of $T$ that has the property that if $x$ and $y$ are in $I[H, V]$, then $|f(x)-f(y)|<1/n$ and $I[H, V]$ is not empty.

For each positive integer $n$, let $K(n)$ denote the union of all the sets of type $A(n)$. $K(n)$ is of type $A(n)$. $K(n)$ covers $X$. For each $U$ in $K(n)$ let $a(U)$ denote a point of $I[K(n), U]$ and $f_n|I[K(n), U]$ denote the constant function, $f_n:I[K(n), U]\to\{f(a(U))\}$. Let $f_n$ denote the function from $X$ into $R$ such that if $U$ is an element of $K(n)$, then $f_n|I[K(n), U]$ is a subset of $f_n$. Suppose $B$ is a nonempty subset of $X$. Let $U(B)$ denote the first element of $K(n)$ that contains a point of $B$. Let $b$ denote a point of $BU(B)$. $b$ is an element of $I[K(n), U(B)]$. If $x$ is an element of $BU(B)$, then $x$ is in $I[K(n), U(B)]$ and $f_n(b)=f_n(x)=f(a(U))$. Therefore $f_n|B$ is constant over $BU(B)$. Therefore $f_n$ is a step-like function.

If $c>0$, there is a positive integer $N$ such that $1/N<c$ and if $n\geq N$ and $x\in X$, then $1/n\leq1/N<c$ and there is a first open set $U$ in $K(n)$ that contains $x$. Therefore $f_n(x)=f(a(U))$. Since $a(U)$ and $x$ are both in $I[K(n), U]$, $|f(a(U))-f(x)|<1/n<c$. Therefore $|f_n(x)-f(x)|<c$. Therefore $f_n$ converges uniformly to $f$. Therefore (a) implies (b).

Proof that (b) implies (c). Since every step-like function has property P, (b) implies (c) directly.

Proof that (c) implies (a). Suppose (c). Let $f_1, f_2, f_3, \cdots$ denote a sequence of functions, each with property P, that converges uniformly to $f$. If $H$ is a nonempty perfect subset of $X$, then $H$ is the union of a countable collection of nonempty perfect subsets of $X$. Therefore, for each positive integer $n$, $f_n|H$ contains a point where $f_n|H$ is continuous. Therefore $f_n$ is of Baire class 1. Therefore $f$ is the uniform limit of a sequence of functions each of Baire class 1. Therefore $f$ is of

Baire class 1 [4, p. 269]. Therefore (c) implies (a). There (a), (b) and (c) are equivalent.

4. EXAMPLE. For each positive integer $n$ and each positive integer $k$ such that $1 \leq k \leq 3^{n-1}$, let $c(n, k)$ denote the Cantor set over the interval $[(3k-2)3^{-n}, (3k-1)3^{-n}]$. For each positive integer $n$, let $f_n$ denote the function from $[0, 1]$ into $[0, 1]$ such that if $x$ is in $c(n, k)$ $- \{(3k-2)3^{-n}, (3k-1)3^{-n}\}$, then $f_n(x) = 1/n$. Otherwise $f_n(x) = 0$. Let $c(0)$ denote the Cantor set over the interval $[0, 1]$. Let $f_0$ denote the function from $[0, 1]$ into $[0, 1]$ such that if $x$ is in $c(0)$, then $f_0(x)$ $= 1$. Otherwise $f_0(x) = 0$.

Let $f$ denote $\sum f_n$. $f$ is of Baire class 1. Let $C$ denote the subset of $[0, 1]$ such that $x \in C$ iff $f(x) \neq 0$. $C$ is the nonempty union of a countable collection of perfect sets. $f | C$ contains no point where $f | C$ is continuous. Therefore $f$ does not have property P. Therefore property P and Baire class 1 are not equivalent.

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