

EMBEDDING FREE ALGEBRAS IN SKEW FIELDS

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ABSTRACT. This paper constructs a minimal element in the partial order on the set of skew fields generated by a free algebra, and shows that the partial order contains a certain sub partial order. Examples of embedding free algebras in skew fields of heights one and two are also given.

Let R be an integral domain (noncommutative) with identity. We will assume that the integral domain R is always embedded in some skew field (this need not be the case in general [2], [3], [6], [7]). There are many skew fields which contain R but in the commutative case there is only one field which is generated by R , namely the field of fractions. In the noncommutative case there may be several distinct skew fields generated by R [4, pp. 277]. For skew fields D_1 and D_2 generated by R , we say $D_1 \geq D_2$ if there exists a place from D_1 to D_2 which extends the natural isomorphism between the embeddings of R . This paper shows that this is a partial order on the set \mathcal{O} (we identify isomorphic embeddings) of skew fields generated by R and in the case where R is the free algebra on two generators we show that \mathcal{O} contains the subposet \mathcal{L} where $C(x)$ is the unique maximal element [1, Theorem 27] of \mathcal{O} and K_i are distinct elements of \mathcal{O} with K_2 minimal in \mathcal{O} . We also examine the height of an integral domain and give examples of embeddings of different heights of a free algebra in two skew fields.

I. The partial order and the chain of domains $Q_i(K, \phi(R))$.

DEFINITION. Let D be a skew field and ϕ an isomorphism of R into D . R is fully embedded in D if the smallest sub skew field of D containing $\phi(R)$ is D itself. We will denote a full embedding of $(D, \phi(R))$ by (D, ϕ) .

DEFINITION. If D_1 and D_2 are two division rings we say ϕ is a place from D_1 to D_2 if ϕ is a homomorphism from a local subring S of D_1 onto D_2 [local means the set of nonunits is an ideal].

DEFINITION. For full embeddings (D, α) and (K, γ) we say $(D, \alpha) \geq (K, \gamma)$ if there is a place ϕ from D to K such that $\phi^{-1}(K) \supset \alpha(R)$ and $\phi|_{\alpha(R)} = \gamma\alpha^{-1}$.

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$(D, \alpha) \cong (K, \gamma)$ if there is an isomorphism ϕ between D and K such that $\phi|_{\alpha(R)} = \gamma\alpha^{-1}$. If we identify isomorphic embeddings the class of embeddings is a set.

We also make the following notation. Suppose ϕ is an isomorphism of R into a skew field K .

$$Q_0(K, \phi(R)) = \phi(R).$$

$Q_1(K, \phi(R)) =$ subring of K generated by $\{a, b^{-1} \text{ such that } a, b \in Q_0(K, \phi(R)) \text{ with } b \neq 0\}$.

Inductively we define

$Q_i(K, \phi(R)) =$ subring of K generated by $\{a, b^{-1} \text{ such that } a, b \in Q_{i-1}(K, \phi(R)) \text{ with } b \neq 0\}$.

When no confusion arises, we will write $Q_i, Q_i(K, \phi)$ or $Q_i(\phi(R))$ for $Q_i(K, \phi(R))$.

It is easy to see that $\cup_i Q_i(K, \phi(R))$ is the skew subfield of K generated by R and $(\cup Q_i, \phi)$ is a full embedding.

LEMMA 1. *The relation \cong is a partial order on the set of full embeddings \mathcal{O} of R .*

PROOF. We will show that if $(D_1, \alpha) \cong (D_2, \gamma)$ and $(D_2, \gamma) \cong (D_1, \alpha)$ that $(D_1, \alpha) \cong (D_2, \gamma)$. The rest is clear. Let ϕ_1 be a place from S_1 onto $D_2, D_1 \supset S_1 \supset \alpha(R)$ and ϕ_2 a place from S_2 onto $D_1, D_2 \supset S_2 \supset \gamma(R)$. $\phi_1|_{\alpha(R)} = \gamma\alpha^{-1} = (\alpha\gamma^{-1})^{-1} = \phi_2^{-1}|_{\alpha(R)}$. Hence $r \neq 0 \in Q_0(D_1, \alpha)$ implies $\phi_1(r) \neq 0 \Rightarrow r^{-1} \in S_1$ so that $S_1 \supset Q_1(D_1, \alpha)$. $\phi_1|_{Q_1(D_1, \alpha)}$ is an isomorphism since $\phi_1(\sum m_i) = 0$, where m_i is a product of elements of $\alpha(R)$ and inverses of elements of $\alpha(R)$, implies $\sum m_i = \phi_2(\sum \phi_2^{-1}(m_i)) = \phi_2(\sum \phi_1(m_i)) = \phi_2[\phi_1(\sum m_i)] = 0$. $\phi_1 = \phi_2^{-1}$ on $Q_1(D_1, \alpha)$. Similarly ϕ_1 is an isomorphism on all Q_i and hence

$$D_1 = \cup Q_i(D_1, \alpha) \stackrel{\phi_1}{\cong} \cup Q_i(D_2, \gamma) = D_2.$$

LEMMA 2. *If (K, α) is a full embedding and $Q_i(K, \alpha)$ is simple for each $i \geq 1$ then (K, α) is minimal in \mathcal{O} .*

PROOF. Suppose $(K, \alpha) \cong (D, \gamma)$. Hence ϕ is a homomorphism from a local subring S of K onto D . $\phi|_{\alpha(R)} = Q_0$ is an isomorphism. Suppose $\phi|_{Q_j}$ is an isomorphism. Hence $Q_{j+1} \subset S$ and ϕ maps $Q_{j+1}(K, \alpha)$ onto $Q_{j+1}(D, \gamma)$. Since $Q_{j+1}(K, \alpha)$ is simple ϕ is an isomorphism on Q_{j+1} . Hence ϕ is an isomorphism from $\cup Q_i = K$ onto D .

Our last general observation is the following. Let I be an ideal of Q_{i+1} , and let i be the inclusion of Q_i into Q_{i+1} . This induces a monomorphism \bar{i} from Q_i into Q_{i+1}/I . This is clear since every element of Q_i has an inverse in Q_{i+1} and hence in Q_{i+1}/I . Q_i and its inverses still generate Q_{i+1}/I .

II. The K -height of a free algebra in two different skew fields.

Suppose ϕ is an isomorphism of R into a skew field K . The K -height of $\phi(R)$ is $h_K(\phi(R)) = \min \{i: Q_i(K, \phi(R)) \text{ is a skew field}\}$ or ∞ if Q_i is not a skew field for any i .

Denote the free algebra on a set X over the field F by $F[X]$. B. H. Neumann [8] gave an example of an embedding of $F[X]$ in a skew field D and conjectured that $h_D(F[X]) = \infty$. Jategaonkar [5] gives a different embedding. We show that this has height 2 and construct another embedding of height one.

Let Q be the rationals and let $F = Q(t_1, t_2, \dots)$ be the commutative field with indeterminates t_1, t_2, \dots . Denote the skew polynomial ring $F[x; \sigma]$ by S where σ is the monomorphism of F into F induced by mapping t_i into t_{i+1} , and where $\sigma(t_i)x = xt_i$. S is embedded in the division ring $K = \{x^{-n} \sum_{i=0}^{\infty} f_i x^i \text{ where } f_i \in F\}$. Since $xS + t_1xS$ is direct, the ring $R = Q[x, t_1x]$ is a free algebra over Q and embedded in K (Jategaonkar [5]).

THEOREM 1. $h_K(R) = 2$.

We will use the following series of lemmas to prove this.

LEMMA 3. $Q_2(R)$ is a skew field.

PROOF. The subring $Q_1(R)$ contains x and x^{-1} and hence contains $t_i = x^{i-1}(t_1x)x^{-i}$. Thus $Q_1(R) \supset Q[t_1, \dots][x; \sigma]$ which is left Ore with a ring of quotients K_1 . Thus taking quotients again we get $Q_2(R) \supset K_1 \supset Q_1(R)$. Since K_1 is a field, by taking quotients again we have $Q_3(R) \supset K_1 \supset Q_2(R)$ and the lemma is proven.

LEMMA 4. If $r \in R$ then $r = \sum f_i x^i$ where $f_i = \sum_I \alpha_I t_1^{\epsilon_1} t_2^{\epsilon_2} \dots t_i^{\epsilon_i}$, $\alpha_I \in Q$, where I ranges over all i -tuples $(\epsilon_1, \dots, \epsilon_i)$ with $\epsilon_j = 1$ or 0 .

PROOF. Since R is a free algebra it is sufficient to examine a monomial m . We claim that if m is of degree i then $m = t_1^{\epsilon_1} t_2^{\epsilon_2} \dots t_i^{\epsilon_i} x^i$ where $\epsilon_j = 1$ or 0 . If $i = 1$ then this is true, since the only monomials we have are αx and $\beta t_1 x$ where $\alpha, \beta \in Q$. Suppose true for $i \leq n$. If m is of degree $n + 1$, then $m = m' t_1^{\epsilon_{n+1}} x$ where m' has degree n , and ϵ_{n+1} is either 1 or 0. By induction $m' = t_1^{\epsilon_1} t_2^{\epsilon_2} \dots t_n^{\epsilon_n} x^n$ and $m = t_1^{\epsilon_1} t_2^{\epsilon_2} \dots t_n^{\epsilon_n} t_{n+1}^{\epsilon_{n+1}} x^{n+1}$. Gathering monomials of same degree we have the lemma.

The following notation will be used in the remaining lemmas. Define $\mathcal{S} = \{ \sum_I \alpha_I t_1^{\epsilon_1} t_2^{\epsilon_2} \dots t_i^{\epsilon_i} \neq 0 \text{ where } I \text{ ranges over all } i\text{-tuples } (\epsilon_1, \dots, \epsilon_i) \text{ where } \epsilon_j = 1 \text{ or } 0, \text{ and } i \text{ ranges over all integers} \}$. Let \mathfrak{M} be all products of elements of \mathcal{S} . It is easy to see that $\sigma(\mathcal{S}) \subset \mathcal{S}$ and thus $\sigma(\mathfrak{M}) \subset \mathfrak{M}$. Since \mathfrak{M} is multiplicatively closed, let $V = Q[t_1, t_2, \dots]$ localized about \mathfrak{M} . $\sigma(V) \subset V$ and from this it is not difficult to see that $U = \{x^{-n} \sum_{i=0}^{\infty} v_i x^i \mid v_i \in V\}$ is a subring of K .

LEMMA 5. $Q_1(R) \subset U$.

PROOF. It is sufficient to show that $r \in R$ implies $r^{-1} \in U$. Let $r = \sum_{i=0}^n f_i x^{i+i_0}$ where $f_0 \neq 0$ and $f_i \in \mathcal{S}$. $r^{-1} = x^{-i_0} (\sum_{i=0}^n f_i x^i)^{-1} = x^{-i_0} \sum_{i=0}^{\infty} b_i x^i$ where $f_0 b_0 = 1$, and $b_n = -f_0^{-1} \sum_{i=1}^n f_i \sigma^i(b_{n-i})$. Since $b_0 = f_0^{-1} \in V$ and $\sigma(V) \subset V$ we have by induction that $b_n \in V$. Thus $r^{-1} \in U$.

LEMMA 6. $(t_1 - t_2^2)^{-1}$ is not contained in U .

PROOF. If $(t_1 - t_2^2)^{-1}$ were in U then it would be in V . Thus $(t_1 - t_2^2)^{-1} = f/g$ where $g \in \mathfrak{M}$ and $f \in Q[t_1, \dots]$. Therefore $g = f(t_1 - t_2^2)$ where $t_1 - t_2^2$ is irreducible and hence divides $g = \pi p_i$ with $p_i \in \mathcal{S}$. This is impossible however since no p_i has terms with square factors.

The theorem now follows since $(t_1 - t_2^2)^{-1} \notin Q_1(R)$ implies $Q_1(R)$ is not a skew field.

We continue with the same notation, and construct an embedding of height one by factoring out a maximal ideal of $Q_1(K, R)$. Let P be the ideal of $Q[t_1, t_2, \dots]$ generated by $t_1 - t_2^2, t_2 - t_3^2, \dots$. It is clear that $\sigma(P) \subset P$. It is also not difficult to see that

$$Q[t_1, \dots] / P \cong Q[\bar{t}_1, \bar{t}_1^{1/2}, \bar{t}_1^{1/4}, \dots]$$

and hence P is a prime ideal. σ extends to an automorphism of $Q[\bar{t}_1, \bar{t}_1^{1/2}, \dots]$ where $\sigma(\bar{t}_1^{1/2^n}) = \bar{t}_1^{1/2^{n+1}}$. In $Q[t_1, \dots] / P$ every element can be written uniquely as $\sum_k \sum_I \alpha_{I,k} t_1^{k} \bar{t}_1^{i_1} \dots \bar{t}_1^{i_n}$ so that $\mathfrak{M} \cap P = \emptyset$. Therefore factoring out the ideal of $Q_1(Q[x, t_1 x])$ generated by P we get the free algebra $Q[x, t_1 x]$ embedded in the Ore domain $Q[\bar{t}_1, \bar{t}_1^{1/2}, \bar{t}_1^{1/4}, \dots][x; \sigma]$ and thus in a skew field, namely the domain's field of quotients K_2 .

We will write t for \bar{t}_1 .

THEOREM 2. $Q_1(K_2, Q[x, tx])$ is a division ring.

PROOF. Since x and x^{-1} are in Q_1 , all integral powers of x are in Q_1 . We first observe that if $p(x) \in Q[t, t^{1/2}, \dots][x; \sigma]$ then there exists integers n and m such that $x^n p(x) x^m \in Q[x, tx]$. Suppose $p(x) = \sum_{i=0}^k a_i x^i$ where

$$a_i = \sum_{j=0}^{k_i} \left(\sum_I \alpha_{I,i,j} t^{\epsilon_1/2} \dots t^{\epsilon_m/2^m} \right) t^j$$

where I runs over the m -tuples $(\epsilon_1, \dots, \epsilon_m)$ with each $\epsilon_i = 0$ or 1 . Pick n to be an integer such that $2^n \geq \max \{k_i\}$.

Therefore $x^n p(x) = \sum_{i=0}^k a'_i x^{i+n}$ where each

$$a'_i = \sum_{j=0}^{k_i} \left(\sum_I \alpha_{I,i,j} t^{\epsilon_1/2^{n+1}} \dots t^{\epsilon_m/2^{n+m}} \right) t^{j/2^n}.$$

Each j can be written in the form $\gamma_n + \gamma_{n-1}2 + \dots + \gamma_0 2^n$ with $\gamma_i = 0$, or 1. Hence

$$t^{j/2^n} t^{\epsilon_1/2^{n+1}} \dots t^{\epsilon_m/2^{n+m}} = t^{\gamma_0} t^{\gamma_1/2} \dots t^{\gamma_n/2^n} t^{\epsilon_1/2^{n+1}} \dots t^{\epsilon_m/2^{n+m}}$$

where $(\gamma_0, \gamma_1, \dots, \gamma_n, \epsilon_1, \dots, \epsilon_m)$ is an $n+m$ tuple of zeros and ones. Hence $x^n p(x) x^m \in Q[x, tx]$.

Since $R = Q[t, t^{1/2}, \dots][x; \sigma]$ is Ore and contained in Q_1 we may assume that an element of Q_1 is written $q(x)^{-1} p(x)$ with $q(x)$ and $p(x) \in R$.

By the above we can choose integers n and m such that both $x^n p(x) x^m$ and $x^n q(x) x^m$ are in $Q[x, tx]$.

Hence $(x^n p(x) x^m)^{-1} \in Q_1$ and

$$1 = (q(x)^{-1} p(x)) [x^m (x^n p(x) x^m)^{-1} (x^n q(x) x^m) x^{-m}] \in Q_1$$

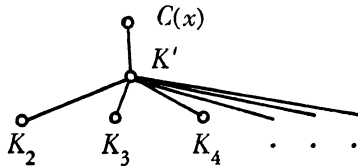
so that Q_1 is a division ring.

COROLLARY 1. $h_{K_1}(Q[x, tx]) = 1$.

III. **The poset of skew fields.** Let K' be the field of quotients of $Q(t_1, t_2, \dots)[x; \sigma]$ where $\sigma(t_i) = t_{i+1}$ and $xf = \sigma(f)x$ for $f \in Q(t_1, t_2, \dots)$. Let F_n be $Q[t_1, t_2, \dots]$ localized about the prime ideal $P_n = \langle t_1 - t_2^n, t_2 - t_3^n, \dots \rangle$ for a positive integer $n > 1$. $\sigma(P_n) \subset P_n$ so we can localize $F_n[x; \sigma]$ about the ideal generated by P_n , to obtain S_n . S_n is a local ring which maps onto the field of quotients K_n of $Q(s, s^{1/n}, s^{1/n^2}, \dots)[y; \sigma]$. If $Q[u, v]$ is the free algebra on two generators, the map $\phi(u) = x$ and $\phi(v) = t_1 x$ maps $Q[u, v]$ onto a free algebra generating K' . Likewise $\phi_n(u) = y$ and $\phi_n(v) = sy$ maps $Q[u, v]$ onto a free algebra generating K_n .

THEOREM 3. $(K_n, \phi_n) \not\geq (K_m, \phi_m)$ for $n \neq m$.

PROOF. The quotient field of $Q(s, s^{1/n}, s^{1/n^2}, \dots)[y; \sigma]$ is K_n where $ys = s^{1/n}y$ and the quotient field of $Q(r, r^{1/m}, r^{1/m^2}, \dots)[z; \sigma]$ is K_m where $zr = r^{1/m}z$. If $(K_n, \phi_n) \geq (K_m, \phi_m)$ and γ is the place then $\gamma(s) = \gamma(sy y^{-1}) = rz z^{-1} = r$ but $\gamma(s^{1/n}) = \gamma(y sy y^{-2}) = zr z z^{-2} = r^{1/m}$. $\gamma[(s^{1/n})^n] = (r^{1/m})^n = r$ which implies $n = m$.



Hence we have $(K', \phi) > (K_n, \phi_n)$ for each $n > 1$. Furthermore, since $Q_1(K_2, \phi_2)$ is a skew field and hence simple, Lemma 2 says (K_2, ϕ_2) is minimal. By [1, Theorem 27], \mathcal{O} has a maximal element $C(x)$ and we have the subposet \mathcal{L} embedded in \mathcal{O} .

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