ON THE AUTOMORPHISM GROUP OF A FINITE p -GROUP WITH THE CENTRAL QUOTIENT METACYCLIC

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ABSTRACT. In this paper it is shown that if G is a finite p -group ($p > 2$) such that the central quotient group of G is metacyclic and nontrivial, then the order of G divides the order of the automorphism group of G .

The purpose of this paper is to extend the known classes of finite p -groups where the order of the group divides the order of the automorphism group. There have been a number of results in this direction [2], [3], [4], [6], [7]. We shall show that this is the case for a finite p -group where the central quotient is metacyclic.

The following notation is used: G is a finite p -group where p is an odd prime; $A(G)$ is the group of automorphisms of G ; $A_c(G)$ and $\text{Inn}(G)$ are the normal subgroups of $A(G)$ of central automorphisms and inner automorphisms, respectively; G_2 is the derived group of G ; $Z(G)$ and $Z_2(G)$ are the center and second center of G , respectively (or Z and Z_2 if no ambiguity is possible); $\exp G$ is the exponent of G ; $|G|$ is the cardinality of the set G and hence the order of the group G ; for x and y in G , $o(x)$ is the order of the element x , (x, y) is the commutator $x^{-1}y^{-1}xy$, x^y is the conjugate $y^{-1}xy$ of x by y , and $\langle x \rangle$ is the cyclic subgroup generated by x ; more generally if S is a subset of G , $\langle S \rangle$ is the subgroup generated by S ; and $\Omega_n(G) = \{x \in G : o(x) \leq p^n\}$. In addition, $H \leq G$ means H is a subgroup of G and $\bigoplus$ is used to denote the direct sum of subgroups. Finally, for groups A and B , $\text{Hom}(A, B)$ is the set of all homomorphisms from A into B .

There are certain results which we often need and shall use throughout the paper without further reference. Their proofs may be found in [5] or [8]. If a, b and c are elements of a group G , then $(a, bc) = (a, c) \cdot (a, b)^c$ and $(ab, c) = (a, c)^b(b, c)$. Also $|\text{Hom}(\bigoplus_i A_i, \bigoplus_j B_j)| = \prod_{i,j} |\text{Hom}(A_i, B_j)|$ where A_i and B_j are Abelian p -groups. In addition, $\text{Hom}(C(p^\alpha), C(p^\beta))$ is isomorphic to $C(p^{\min(\alpha, \beta)})$ where $C(n)$ is the cyclic group of order n . Finally, it is assumed that the definition of a regular p -group and the basic properties of such groups are known.

We now state our theorem.

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THEOREM. *If G is a finite p -group ($p > 2$) such that G/Z is metacyclic and nontrivial, then $|G|$ divides $|A(G)|$.*

We may assume that the nilpotency class of G is greater than 2 [4]. Furthermore, we may assume G is purely non-Abelian [6] and consequently that $|A_c(G)| = |\text{Hom}(G, Z)|$ [1].

Since G/Z is metacyclic, there is a normal subgroup B of G/Z such that B and $(G/Z)/B$ are both cyclic. Consequently, we can choose a and b in G such that $B = \langle bZ \rangle$ and $(G/Z)/B = \langle (aZ)B \rangle$. Let $M = \langle b, Z \rangle$ and $o(bZ) = p^m$. Then M is a normal Abelian subgroup of G where G/M is cyclic generated by aM . Furthermore, G_2 is a subgroup of M and so b commutes with every commutator. Consequently, we can easily show by using induction and the basic commutator identities that $(a, b^n) = (a, b)^n$ for every positive integer n . Then since $o(bZ) = p^m$, we have that $(a, b)^{p^m} = (a, b^{p^m}) = e$. Furthermore, since $G = \langle a, b, Z \rangle$, $(a, b)^{p^n} = (a, b^{p^n}) \neq e$ when $0 < n < m$. So $o(a, b) = p^m$.

It can be shown that the map $M \rightarrow G_2$ defined by $g \mapsto (g, a)$ is a homomorphism of M onto G_2 with kernel $M \cap Z$ [5, Aufgabe 2, S. 259]. Thus $|M| = |G_2| \cdot |M \cap Z| = |G_2| \cdot |Z|$.

Since $M = \langle b, Z \rangle$ where $o(bZ) = p^m$, we have $|G_2| = p^m$. Since (a, b) is in G_2 with order p^m , $G_2 = \langle (a, b) \rangle$ and hence G is regular [5, Satz 10.2]. Because G is regular, $e = (a, b)^{p^m} = (a^{p^m}, b)$ and so a^{p^m} is in Z . As above we see that a^{p^n} is not in Z when $0 < n < m$. So $o(aZ) = p^m$.

We now claim that we can choose a such that $(a, b) = b^{p^\alpha}y$ where $0 \leq \alpha < m$ and y is in Z . First, since (a, b) is in $G_2 \leq M = \langle b \rangle Z$, we have $(a, b) = b^{rp^\alpha}x$ where $(r, p) = 1$ and x is in Z . Furthermore, since $o(bZ) = p^m$ and the nilpotency class of G is greater than 2, we choose α so that $0 \leq \alpha < m$. Because $o(aZ) = p^m$, $\{(a^s, b) : 0 \leq s < p^m\}$ is a set with p^m distinct elements and hence is equal to G_2 . Because $(r, p) = 1$, we can find n so that $(n, p) = 1$ and $rn \equiv 1 \pmod{o(b)}$. Since $(b^{rp^\alpha}x)^n = b^{p^\alpha}x^n$ is a generator of G_2 , we can choose s , $0 \leq s < p^m$, such that $(a^s, b) = b^{p^\alpha}x^n$. Furthermore, $(s, p) = 1$ since otherwise the regularity of G would imply that $o(a^s, b) < p^m$. Finally, since $(s, p) = 1$, $\langle (a^sZ)B \rangle = \langle (a^sZ)B \rangle$ and our claim is established.

Let us also observe that $o((a, b)Z) = o(b^{p^\alpha}yZ) = p^{m-\alpha}$. Thus $|G_2Z| = p^{m-\alpha}|Z|$ and since $|G_2| = p^m$, $|G_2 \cap Z| = p^\alpha$. Also since $Z \leq M$, we have $p^m = o(aZ) \geq o(aM)$. On the other hand, whenever a^{p^n} is in M , we have $e = (a^{p^n}, b) = (a, b)^{p^n}$ and so $o(aM) \geq p^m$. Consequently, $G/M = \langle aM \rangle$ is cyclic of order p^m . It now follows that $|G/Z| = |G/M| \cdot |M/Z| = p^{2m}$ and so $|\text{Inn}(G)| = p^{2m}$.

We now seek a suitable lower bound on $|A_c(G)|$. This will be done by examining $|\text{Hom}(G, Z)|$ which is the same as $|\text{Hom}(G/G_2, Z)|$ [8, 3.4.19]. Let $p^k = \exp G/G_2$ and $p^l = \exp Z$. Since $\exp Z = p^l$, we have

$Z = \langle z_1 \rangle \oplus Z_1$ where $o(z_1) = p^l$ and $\exp Z_1 = p^\beta$, $\beta \leq l$. Since $G_2 \leq M$ and $o(aM) = p^m$, $o(aG_2) \geq p^m$ and so $k \geq m$.

LEMMA 1. $|A_c(G)| \geq p^{\alpha+\beta} |Z|$ or $p^{2\alpha} |Z|$.

Case 1. $l \geq k$. Let $Z_1 = (\bigoplus_{i=1}^n \langle z'_i \rangle) \oplus Z'_1$ where $o(z'_i) \geq p^k$ and $\exp Z'_1 < p^k$. Then similar to Adney and Yen [1], we have

$$\begin{aligned} |\operatorname{Hom}(G/G_2, Z)| &= |\operatorname{Hom}(G/G_2, \langle z_1 \rangle)| \\ &\quad \cdot \prod_{i=1}^n |\operatorname{Hom}(G/G_2, \langle z'_i \rangle)| \cdot |\operatorname{Hom}(G/G_2, Z'_1)| \\ &\geq |G/G_2|^{n+1} |Z'_1|. \end{aligned}$$

If $|A_c(G)| \geq |G| = p^{2m} |Z|$, then $|A_c(G)| \geq p^{2\alpha} |Z|$ and we are done. Thus we assume $|G| > |A_c(G)|$. If $n > 0$, then we have

$$|G/G_2| p^m = |G| > |A_c(G)| \geq |G/G_2|^{n+1} |Z'_1| \geq |G/G_2|^{n+1}$$

and so

$$p^m > |G/G_2|^n \geq |G/G_2| \geq \exp G/G_2 = p^k,$$

which is a contradiction. Thus $n=0$ and $Z'_1 = Z_1$. Finally, $|A_c(G)| \geq |G/G_2| |Z_1| \geq (|G|/|G_2|) \exp Z_1 = p^m |Z| p^\beta \geq p^{\alpha+\beta} |Z|$.

Case 2. $k > l$. Let $G/G_2 = \bigoplus_{i=1}^t \langle \bar{b}_i \rangle$ where $\bar{b}_i = b_i G_2$ with b_i in G and $p^k = o(\bar{b}_1) \geq o(\bar{b}_2) \geq \dots \geq o(\bar{b}_t)$. We note that $t \geq 2$. We have that $|\operatorname{Hom}(G/G_2, Z)| = |\operatorname{Hom}(\langle \bar{b}_1 \rangle, Z)| \prod_{i=2}^t |\operatorname{Hom}(\langle \bar{b}_i \rangle, Z)|$. Since $o(\bar{b}_1) = p^k > \exp Z$, $|\operatorname{Hom}(\langle \bar{b}_1 \rangle, Z)| = |Z|$. If $o(\bar{b}_2) \geq p^l$, then $\prod_{i=2}^t |\operatorname{Hom}(\langle \bar{b}_i \rangle, Z)| \geq |\operatorname{Hom}(\langle \bar{b}_2 \rangle, Z)| = |Z| \geq p^l |Z_1| \geq p^{l+\beta} \geq p^{\alpha+\beta}$, since $Z \cap G_2$ is a cyclic subgroup of order p^α in Z where $\exp Z = p^l$. Thus we shall assume $o(\bar{b}_2) < p^l$.

We now claim that $o(\bar{b}_2) \geq p^\alpha$. Since $G = \langle \{b_1, \dots, b_t\} \rangle$, we have that G_2 is the normal closure of the set $\{(b_i, b_j) : 1 \leq i, j \leq t\}$ [8, 3.4.18]. Since G is regular, $\exp G_2 \leq \max \{o(b_i, b_j) : 1 \leq i, j \leq t\}$. But if $o(\bar{b}_2) < p^\alpha$, $b_i^{p^{\alpha-1}}$ is in G_2 for $i \geq 2$. Since $|G_2 Z / Z| = |G_2 / G_2 \cap Z| = p^{m-\alpha}$, $(b_i^{p^{\alpha-1}})^{p^{m-\alpha}} = b_i^{p^{m-1}}$ is in Z for $i \geq 2$. Hence $o(b_i, b_j) \leq p^{m-1}$ for i, j where $1 \leq i, j \leq t$ and so $\exp G_2 \leq p^{m-1}$, a contradiction. So our claim is established.

Now

$$\begin{aligned} \prod_{i=2}^t |\operatorname{Hom}(\langle \bar{b}_i \rangle, Z)| &= \prod_{i=2}^t |\operatorname{Hom}(\langle \bar{b}_i \rangle, \langle z_1 \rangle)| \prod_{i=2}^t |\operatorname{Hom}(\langle \bar{b}_i \rangle, Z_1)| \\ &\geq \prod_{i=2}^t o(\bar{b}_i) |\operatorname{Hom}(\langle \bar{b}_2 \rangle, Z_1)| \geq p^\alpha |\operatorname{Hom}(\langle \bar{b}_2 \rangle, Z_1)|. \end{aligned}$$

Furthermore, $|\text{Hom}(\langle \bar{b}_2 \rangle, Z_1)| \geq p^\beta$ or p^α when $o(\bar{b}_2) \geq p^\beta$ or $o(\bar{b}_2) < p^\beta$, respectively. Thus the proof of the lemma is complete.

We now seek to determine a lower bound on $|R|$, where R is the normal subgroup $\text{Inn}(G)A_c(G)$ of $A(G)$. For this we determine $|\text{Inn}(G) \cap A_c(G)| = |Z_2/Z|$.

LEMMA 2. $|Z_2/Z| = p^{2\alpha}$.

For purposes of notation, let $H = G/Z$, $\bar{a} = aZ$, and $\bar{b} = bZ$. Then $o(\bar{a}) = o(\bar{b}) = p^m$, $\langle \bar{b} \rangle$ is normal in H , and $H/\langle \bar{b} \rangle = \langle \bar{a}\bar{b} \rangle$ is cyclic of order p^m . So H_2 is cyclic generated by $(\bar{a}, \bar{b}) = \bar{b}^{p^\alpha}$ which has order $p^{m-\alpha}$. Using the regularity of H , we have as before that $o(\bar{a}Z(H)) = o(\bar{b}Z(H)) = p^{m-\alpha}$. Furthermore, since $|H| = p^{2m}$ and $H = \langle \bar{a} \rangle \langle \bar{b} \rangle$, we see that $\langle \bar{a} \rangle \cap \langle \bar{b} \rangle = E$. Thus $\langle \bar{a}^{p^{m-\alpha}} \rangle \oplus \langle \bar{b}^{p^{m-\alpha}} \rangle$ is a subgroup of $Z(H)$. Conversely, suppose $\bar{a}^r \bar{b}^s$ is in $Z(H)$. Then $e = (\bar{a}^r \bar{b}^s, \bar{b}) = (\bar{a}^r, \bar{b})^{\bar{b}^s} (\bar{b}^s, \bar{b}) = (\bar{a}^r, \bar{b})$, since $H_2 = \langle \bar{b}^{p^\alpha} \rangle$. So $\bar{a}^r$ is in $Z(H)$ and $r \equiv 0 \pmod{p^{m-\alpha}}$. Likewise, $\bar{b}^s$ is in $Z(H)$ and so $s \equiv 0 \pmod{p^{m-\alpha}}$. Hence $Z(H) = \langle \bar{a}^{p^{m-\alpha}} \rangle \oplus \langle \bar{b}^{p^{m-\alpha}} \rangle$. It now follows that $|Z_2/Z| = p^{2\alpha}$ as was to be shown.

Now $|R| = |\text{Inn}(G)| |A_c(G)| / |\text{Inn}(G) \cap A_c(G)| = p^{2m-2\alpha} |A_c(G)|$. If $|A_c(G)| \geq p^{2\alpha} |Z|$, then $|R| \geq |G|$ and we are done. Thus we assume $|A_c(G)| \geq p^{\alpha+\beta} |Z|$ where $\beta < \alpha$. Consequently, $|R| \geq p^{2m-\alpha+\beta} |Z|$.

We now observe that z_1 and b may be chosen such that $a^{p^m} \equiv z_1^{p^{t_a}}$ (mod Z_1) and $b^{p^m} \equiv z_1^{p^{t_b}}$ (mod Z_1) where $0 \leq t_a, t_b \leq l$. Since a^{p^m} is in Z , $a^{p^m} = z_1^{p^{t_a}} z_2$ where $0 \leq t_a \leq l$, z_2 is in Z_1 , and $(r, p) = 1$. We replace z_1 by z_1^r and since $\langle z_1 \rangle = \langle z_1^r \rangle$, the first part is done. Likewise, since b^{p^m} is in Z , $b^{p^m} = z_1^{p^{t_b}} z_3$ where $0 \leq t_b \leq l$, z_3 is in Z_1 , and $(s, p) = 1$. There is u where $su \equiv 1 \pmod{o(z_1)}$ and $(u, p) = 1$. Then

$$(b^u)^{p^m} = z_1^{usp^{t_b}} z_3 \equiv z_1^{p^{t_b}} \pmod{Z_1} \quad \text{and} \quad \langle b^u Z \rangle = \langle bZ \rangle.$$

So we replace b by b^u .

In order to complete the proof, we will find it necessary to construct certain automorphisms. The automorphisms are described in the following lemma [2, Lemma 4].

LEMMA 3. Suppose K is a normal subgroup of G where $G/K = \langle cK \rangle$ is cyclic of order p^n . If x is in $\Omega_n(Z(K))$, then the mapping $\phi(c, K, x)$ defined by $c^j y \rightarrow (cx)^j y$, where $0 \leq j < p^n$ and y is in K , is an automorphism of G under which K is elementwise fixed. Furthermore, $o(\phi(c, K, x)) = o(x)$.

We now proceed to the last stage of the proof. Let F be G_2Z . Then $F = \langle b^{p^\alpha} \rangle Z$ and is a normal subgroup of order $p^{m-\alpha} |Z|$. We now divide the proof into two cases.

Case I. $t_a \geq t_b$. Then there is $r \geq 0$ where $t_a = t_b + r$. Let $\bar{a} = b^{-r}a$ and $K = \langle \bar{a}, F \rangle$ which is normal in G . First of all, $(\bar{a}, b) = (a, b)$. Secondly, whenever $\bar{a}^{p^n}$ is in F , $(\bar{a}^{p^n}, b) = e$ and so $(\bar{a}, b)^{p^n} = e$. Thus $p^m \leq o(\bar{a}F)$. On the other hand, $(\bar{a})^{p^m} = (b^{-r}a)^{p^m} = b^{-rp^{r+m}}a^{p^m}x^{p^m}$ with x in G_2 . So $(\bar{a})^{p^m} = b^{-rp^{r+m}}a^{p^m}$ is in $Z \leq F$. Hence $o(\bar{a}F) = p^m$ and $|K| = p^{2m-\alpha}|Z|$. So G/K has order p^α and is cyclic generated by bK . Also $(\bar{a})^{p^m} = b^{-rp^{r+m}}a^{p^m} = z_1^{-p^t b^t} z_1^{p^t a} w$ for some w in Z_1 . Since $t_a = t_b + r$ and $\exp Z_1 = p^\beta$, we have $p^m \leq o(\bar{a}) \leq p^{m+\beta}$. Since $\alpha < m$ we can choose s where $o(\bar{a}^{p^s}) = p^\alpha$ and $m + \beta - \alpha \geq s \geq m - \alpha$. Because $e = (\bar{a}, b)^{p^m} = (\bar{a}^{p^{m-\alpha}}, b^{p^\alpha})$, $\bar{a}^{p^{m-\alpha}}$ is in $Z(K)$ and so $\bar{a}^{p^s}$ is in $\Omega_\alpha(Z(K))$. Then by Lemma 3, $\phi = \phi(b, K, \bar{a}^{p^s})$ is an automorphism of order p^α . Since $o(\bar{a}F) = p^m$, $o(\bar{a}^{p^s}F) = p^{m-s} \geq p^{\alpha-\beta}$. Hence $b^{-1}(b\phi^{p^{(\alpha-\beta)-1}}) = \bar{a}^{p^{s+(\alpha-\beta)-1}}$ is not in $F = G_2Z$ and so $o(\phi R) \geq p^{\alpha-\beta}$. Thus $|\langle \phi, R \rangle| \geq p^{\alpha-\beta}|R| \geq |G|$.

Case II. $t_b > t_a$. In this case we can find $r \geq 1$ where $t_b = t_a + r$. Let $\bar{b} = a^{-r}b$ and $L = \langle \bar{b}, F \rangle$ which is normal in G . We note that since $(a, b) = (a, \bar{b})$, $o(\bar{b}) \geq p^m$.

Subcase (i). $r + \alpha \geq m$. We first note that G/L is a cyclic group generated by aL . Since $Z \leq L$ and $o(aZ) = p^m$, $o(aL) = |G/L| \leq p^m$. On the other hand, $\bar{b}^{p^\alpha} = (a^{-r}b)^{p^\alpha} = a^{-rp^{r+\alpha}}b^{p^\alpha}g^{p^\alpha}$ with g in $G_2 \leq F$. Since $r + \alpha \geq m$, $a^{-rp^{r+\alpha}}$ is in $Z \leq F$ and so $\bar{b}^{p^\alpha}$ is in F . Thus $|L| \leq p^\alpha|F| = p^m|Z|$. From this we see that $|G/L| \geq p^m$. Hence $|G/L| = p^m$ and $o(\bar{b}F) = p^\alpha$. As in the previous case, we see that $\bar{b}^{p^s}$ is in Z_1 and so $p^m \leq o(\bar{b}) \leq p^{m+\beta}$. Thus we can choose s where $o(\bar{b}^{p^s}) = p^m$ and $0 \leq s \leq \beta$. Also since $r + \alpha \geq m$, $(a^{-1}, b)^{p^{r+\alpha}} = e$ and so $(a^{-r}, b^{p^\alpha}) = e$. Thus $(\bar{b}, b^{p^\alpha}) = e$ and $\bar{b}$ is in $Z(L)$. So finally $\bar{b}^{p^s}$ is in $\Omega_m(Z(L))$. Again by Lemma 3, $\phi = \phi(a, L, \bar{b}^{p^s})$ is an automorphism of order p^m . Since $o(\bar{b}F) = p^\alpha$, $o(\bar{b}^{p^s}F) = p^{\alpha-s} \geq p^{\alpha-\beta}$ and so as before $o(\phi R) \geq p^{\alpha-\beta}$. Thus $|\langle \phi, R \rangle| \geq p^{\alpha-\beta}|R| \geq |G|$.

Subcase (ii). $r + \alpha < m$. We choose $\gamma \geq 1$ such that $r + \alpha + \gamma = m$. First of all, $(\bar{b})^{p^{\alpha+\gamma}} = (a^{-r}b)^{p^{\alpha+\gamma}} = a^{-rp^{r+\alpha+\gamma}}b^{p^{\alpha+\gamma}}g^{p^{\alpha+\gamma}}$ with g in $G_2 \leq F$. Thus $o(\bar{b}F) \leq p^{\alpha+\gamma}$. If $o(\bar{b}F) < p^{\alpha+\gamma}$, then $\bar{b}^{p^{\alpha+\gamma-1}}$ is in F . But $\bar{b}^{p^{\alpha+\gamma-1}} = a^{-rp^{m-1}}(b^{p^\alpha})^{p^{\gamma-1}}g^{p^{\alpha+\gamma-1}}$ where g is in $G_2 \leq F$. Thus we have that $a^{-rp^{m-1}}$ is in $F = \langle b^{p^\alpha} \rangle Z$. Hence $e = (a^{p^{m-1}}, b) = (a, b)^{p^{m-1}}$, a contradiction. So $o(\bar{b}F) = p^{\alpha+\gamma}$. Then $|L| = p^{\alpha+\gamma}|F| = p^{m+\gamma}|Z|$. Consequently, $|G/L| = p^{m-\gamma}$ and G/L is cyclic generated by aL . Again we have $p^m \leq o(\bar{b}) \leq p^{m+\beta}$ and so $p^{m-\gamma} \leq o(\bar{b}^{p^\gamma}) \leq p^{m+\beta-\gamma}$. We can choose s where $o(\bar{b}^{p^s}) = p^{m-\gamma}$ and $\gamma \leq s \leq \gamma + \beta$. We now note that $e = (a^{-1}, b)^{p^m} = (a^{-p^{m-\alpha}}, b^{p^\alpha}) = (a^{-p^{r+\gamma}}, b^{p^\alpha})$. Thus $(\bar{b}^{p^\gamma}, b^{p^\alpha}) = e$ and $\bar{b}^{p^\gamma}$ is in $Z(L)$. So finally $\bar{b}^{p^s}$ is in $\Omega_{m-\gamma}(Z(L))$.

Again, we see that $\phi = \phi(a, L, \bar{b}^{p^s})$ is an automorphism of order $p^{m-\gamma}$. Since $o(\bar{b}F) = p^{\alpha+\gamma}$, $o(\bar{b}^{p^s}F) = p^{\alpha+\gamma-s} \geq p^{\alpha-\beta}$. As before $o(\phi R) \geq p^{\alpha-\beta}$ and so $|\langle \phi, R \rangle| \geq p^{\alpha-\beta}|R| \geq |G|$.

COROLLARY [2]. If G is a noncyclic metacyclic p -group of order greater than p^2 with $p > 2$, then $|G|$ divides $|A(G)|$.

REFERENCES

1. J. E. Adney and T. Yen, *Automorphisms of a p -group*, Illinois J. Math. 9 (1965), 137–143. MR 30 #2072.
2. R. M. Davitt, *The automorphism group of a finite metacyclic p -group*, Proc. Amer. Math. Soc. 25 (1970), 876–879.
3. ———, *The automorphism group of finite p -Abelian p -groups*, Illinois J. Math. (to appear).
4. R. Faudree, *A note on the automorphism group of a p -group*, Proc. Amer. Math. Soc. 19 (1968), 1379–1382. MR 40 #1476.
5. B. Huppert, *Endliche Gruppen*. I, Die Grundlehren der math. Wissenschaften, Band 134, Springer-Verlag, Berlin and New York, 1967. MR 37 #302.
6. A. D. Otto, *Central automorphisms of a finite p -group*, Trans. Amer. Math. Soc. 125 (1966), 280–287. MR 34 #4362.
7. R. Ree, *The existence of outer automorphisms of some groups*. II, Proc. Amer. Math. Soc. 9 (1958), 105–109. MR 20 #912.
8. W. R. Scott, *Group theory*, Prentice-Hall, Englewood Cliffs, N. J., 1964. MR 29 #4785.

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