COMPACT GROUP TOPOLOGIES FOR R

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ABSTRACT. There is a compact solenoidal group S that is group isomorphic to the additive real numbers R . The existence of S leads to a study of compact real groups; that is, compact groups which are group isomorphic to R . The compact real groups are characterized as products of S . Various conditions on a topological group G are given which are equivalent to G being compact real. It is shown that any compact real group is solenoidal, connected and separable. Many sets and functions on R which are measurable with respect to the usual topology are shown to be nonmeasurable with respect to any compact group topology.

1. Introduction. If the α -adic solenoid, herein denoted by S , with $\alpha = (2, 3, 4, \dots)$ is as in §10 of [4], it is known that S is group isomorphic to R , the additive real numbers [2]. S is a compact group that is connected, solenoidal and metrizable (§10 of [4]). This paper characterizes all compact real groups (i.e., compact groups that are isomorphic to R) in terms of S , gives some of their topological properties and relates measurable sets in R_u (R with the usual topology) and R_c (R with a compact group topology).

2. A characterization of compact real groups. For G a topological group, G^r denotes the direct product of r copies of G with the product topology and $\hat{G}$ denotes the topological dual of G . T denotes the complex numbers of modulus 1, and Z denotes the integers.

THEOREM 2.1. *G is a compact real group if and only if G is topologically isomorphic to S^r where $2 \leq 2^r \leq c$ with $c = \text{card}(R)$.*

PROOF. By (25.33) of [4] it follows that a compact topological group is torsion-free and divisible if and only if G is topologically isomorphic to S^r for some cardinal r . A torsion-free, divisible group is isomorphic to R if and only if it is of cardinal c . Therefore, it suffices to show $\text{card}(S^r) = c$ if and only if $2 \leq 2^r \leq c$.

Since $\text{card}(S^r) = c^r = (2^{\aleph_0})^r = 2^{\aleph_0 r}$ and $2^{\aleph_0 r} = c$ if and only if $2 \leq 2^r \leq c$, $\text{card}(S^r) = c$ if and only if $2 \leq 2^r \leq c$.

For $r \neq p$, S^r is not topologically isomorphic to S^p so $\{S^r : 2 \leq 2^r \leq c\}$ is a canonical set of representatives for the set of compact real groups under the equivalence relationship of topological isomorphism. If the

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continuum hypothesis is assumed, the condition " $2 \leq 2^r \leq c$ " may be replaced by " $1 \leq r \leq \aleph_0$ ".

If $2 \leq 2^r \leq 2^p \leq c$ and $r < p$, then the topological isomorphism of S^r into S^p that identifies S^r with r factors of S^p is onto a proper compact real subgroup of S^p . Each subgroup of $S^{\aleph_0}$ composed of denumerably many factors of S is topologically isomorphic to $S^{\aleph_0}$.

From what is known about the structure of compact groups (§24 of [4]) and Theorem 2.1, the following corollary may be seen.

COROLLARY 2.1. *If the continuum hypothesis is assumed, the following are equivalent for a nontrivial compact abelian group G :*

- (i) G is compact real.
- (ii) G is metrizable and connected and $G^\wedge$ is divisible.
- (iii) G is metrizable and $G^\wedge$ is torsion-free and divisible.
- (iv) G is metrizable, connected and torsion-free.

Without the assumption of the continuum hypothesis, the following are equivalent for a nontrivial metrizable compact abelian group G :

- (i) G is compact real.
- (ii) G is connected and $G^\wedge$ is divisible.
- (iii) $G^\wedge$ is torsion-free and divisible.
- (iv) G is connected and torsion-free.

3. Properties of a compact real group R_c . There are some elementary similarities between R with a compact topology and R with the usual topology.

THEOREM 3.1. *A compact real group R_c is solenoidal, connected and separable. Assuming the continuum hypothesis, R_c is metrizable.*

PROOF. Since R_c is the product of r copies of S where $r < c$, R_c is solenoidal (§25 of [4]). R_c is connected and separable since S is. Assuming the continuum hypothesis, R_c is the product of countably many metrizable groups and as such is metrizable.

Since R with either a compact topology or the usual topology is connected, both R_c and R_u have no proper open subgroups. However, R_u has proper open semigroups (e.g., $\{x: x > 0\}$), but no R_c has a proper semigroup with nonvoid interior since every R_c is compact and connected (see [1]).

The compactness of R_c insures that it has no nontrivial countable subgroups. Some information about subgroups of R_c may be gained from the structure of S . In S there are dense divisible Borel subgroups of cardinality c whose index in S is c . For instance, if S is expressed as $R_u \times \Delta_a / B$, where Δ_a and B are as in §10 of [4], $Q \times \Delta_a / B$, where Q is the rational numbers, has these properties. Any divisible subgroup of

R of cardinality c is group isomorphic to R . This leads to the following theorem.

THEOREM 3.2. *If R_C is a compact real group, R_C contains a dense Borel subgroup H isomorphic to R whose index in R_C is c (and hence has Haar measure 0). If further $R_C^\wedge$ is not topologically isomorphic to S , R_C contains a proper subgroup J which is compact real and whose index in R_C is c (and hence has Haar measure 0).*

PROOF. Let K be a dense Borel subgroup of S that is isomorphic to R and has index c in S . By Theorem 2.1, R_C is topologically isomorphic to S^r where $2 \leq 2^r \leq c$. Consider $H = K \times S^{r-1}$ (where $r-1 = r$ if r is infinite). This subgroup satisfies the first part of the theorem.

The second part of the theorem follows from the remark following Theorem 2.1.

4. Measurable sets in R_C and R_u . In the following R denotes the additive reals with no topology, R_u and R_C have the same meaning as above. As might be expected, the usual topology for R is quite different from any compact group topology. None of the nontrivial continuous characters on R_u are continuous on R_C .

THEOREM 4.1. *Say $\chi_y: R_C \rightarrow T$ is defined by $\chi_y(x) = e^{ixy}$ for $y \in R$. If $y \neq 0$, χ_y is not continuous.*

PROOF. Assume χ_y is continuous for some $y \neq 0$. Since χ_y is a homomorphism, $\chi_y \in R_C^\wedge$.

Since R_C is torsion-free, $R_C^\wedge$ is divisible. So, for every $n \in \{1, 2, 3, \dots\}$, there exists $\Psi_n \in R_C$ such that $(\Psi_n)^{2^n} = \chi_y$. This implies $\Psi_n(x) = \exp(i2^{-n}(xy + 2\pi K(x, n)))$, where $K(x, n) \in \{0, 1, 2, \dots, 2^n - 1\}$, for $x \in R_C$, $n \in \{1, 2, 3, \dots\}$. Thus, for $x \in R_C$, $n \in \{1, 2, 3, \dots\}$,

$$\begin{aligned} \exp(i2^{-n}(xy + 2\pi K(x, n))) &= (\Psi_n(2^{-n}x))^{2^n} \\ &= \exp(i2^{-n}(2^{-n}xy + 2\pi K(2^{-n}x, n)))^{2^n} \\ &= \exp(i2^{-n}(xy + 2^n 2\pi K(2^{-n}x, n))), \end{aligned}$$

so $2^n K(2^{-n}x, n) \equiv K(x, n) \pmod{2^n}$. Since $K(x, n) \in \{0, 1, 2, \dots, 2^n - 1\}$ and $2^n K(2^{-n}x, n) \equiv 0 \pmod{2^n}$, $K(x, n) = 0$. That is, $\Psi_n = \chi_{2^{-n}y}$.

Consider the subgroup G of $R_C^\wedge$ generated by $\{\chi_{2^{-n}y}: n = 0, 1, \dots\}$. G separates points of R_C , hence $G = R_C^\wedge$. But G is isomorphic to the dyadic rationals and is not divisible. This contradicts the fact that $R_C^\wedge$ is divisible. Thus χ_y cannot be continuous for $y \neq 0$.

COROLLARY 4.1. χ_y is not measurable (with respect to R_C -Haar measure) on R_C for $y \neq 0$.

PROOF. If χ_y were R_C -measurable, it would be R_C -continuous ((22.19) of [4]).

COROLLARY 4.2. For nonzero $\alpha \in R$, $\cos(\alpha x)$, $\sin(\alpha x)$, $\tan(\alpha x)$, $\cot(\alpha x)$, $\sec(\alpha x)$ and $\csc(\alpha x)$ are nonmeasurable on R_C .

PROOF. Since $\cos(\alpha x)$ and $\sin(\alpha x)$ are translates of each other, $\cos(\alpha x)$ is measurable if and only if $\sin(\alpha x)$ is. If $\cos(\alpha x)$ and $\sin(\alpha x)$ were measurable, $\exp(i\alpha x) = \cos(\alpha x) + i \sin(\alpha x)$ would be measurable, but this is impossible by Corollary 4.1.

The nonmeasurability of $\tan(\alpha x)$, $\cot(\alpha x)$, $\sec(\alpha x)$ and $\csc(\alpha x)$ on R_C follows from trigonometric identities and properties of measurable functions; e.g., the measurability of $\sec(\alpha x)$ would imply the measurability of $1/\sec(\alpha x) = \cos(\alpha x)$.

From this it is possible to prove that a large class of subsets of R is not R_C -measurable.

LEMMA 4.1. For nonzero $d \in R$, there exists a nonmeasurable set of the form $]a, b[+ dZ$ in R_C .

PROOF. $(\sin(2\pi/d))^{-1}(]t, \infty[)$ is a nonmeasurable subset of R_C for some $t \in R$ since $\sin(2\pi x/d)$ is nonmeasurable on R_C . This set is of the form $]a, b[+ dZ$.

THEOREM 4.2. For nonzero $d \in R$, any set of the form $]a, b[+ dZ$, where $0 < b - a < d$, is nonmeasurable in R_C .

PROOF. By Lemma 4.1, given d , some set $]a', b'[+ dZ$ is nonmeasurable in R_C . If $b' - a' = 0$ or $b' - a' \geq d$, $]a', b'[+ dZ$ is countable or cocountable. In either case, $]a', b'[+ dZ$ is measurable in R_C , so $0 < b' - a' < d$. $]a', b'[+ dZ$ may be derived from any set of the form $]a, b[+ dZ$, where $0 < b - a < d$, by some combination of translation and intersection and union of the translates. All these operations preserve measurability, so if $]a, b[+ dZ$ with $0 < b - a < d$ were measurable in R_C , $]a', b'[+ dZ$ would be.

The following lemma leads to more R_C -nonmeasurable sets.

LEMMA 4.2. In a compact group G , a set $A \subset G$ with the following properties is not Haar measurable:

- (i) there exists $\{x_n\}_{n=1}^{\infty} \subset G$ such that $(x_n + A) \cap (x_m + A) = \emptyset$ for $n \neq m$, and
- (ii) there exists $\{y_n\}_{n=1}^{\infty} \subset G$ such that $G = \bigcup_{n=1}^{\infty} \{y_n + A\}$.

PROOF. Say A satisfies (i) and (ii). If A were measurable, A would have measure 0 by (i), but by (ii), A would have positive measure.

THEOREM 4.3. *A set $A \subset R$ that is bounded in the R_u -metric and has nonvoid R_u -interior is not measurable in R_C .*

PROOF. Any such set satisfies (i) and (ii) of Lemma 4.2.

COROLLARY 4.3. *If A is a finite union of nondegenerate intervals in R such that A' is also a finite union of nondegenerate intervals, then A is nonmeasurable in R_C .*

PROOF. If A is bounded in the R_u -metric, it satisfies the hypotheses of Theorem 4.3 and is nonmeasurable in R_C . If A contains an interval of the form $]-\infty, b[$ and an interval of the form $]a, \infty[$, A' satisfies the hypotheses of Theorem 4.3 and is nonmeasurable in R_C , so A is nonmeasurable in R_C .

Suppose A contains $]-\infty, b[$, but no interval of the form $]a, \infty[$. If A were measurable in R_C , $(A + (-2b+1)) \cap (-A)$ would be measurable in R_C , but this would contradict Theorem 4.3. Similarly, if A contains $]a, \infty[$, but no interval of the form $]-\infty, b[$, A is nonmeasurable in R_C .

Corollary 4.3 leads to the nonmeasurability of certain functions on R_C .

COROLLARY 4.4. *If f is a nonconstant function defined on R and f satisfies either of the following two conditions, f is not measurable on R_C :*

- (i) *f is R_u -continuous, f is not constant on any nondegenerate interval, and f attains relative maximums at at most finitely many points, or*
- (ii) *f is monotonic.*

PROOF. If f satisfies (i), there are $x_1, x_2 \in R$ so that $f(x_1) > f(x_2)$. Pick $\alpha \in]f(x_2), f(x_1)[$ so α is not a relative minimum or maximum. $f^{-1}(] \alpha, \infty[)$ is then a set of the form described in Corollary 4.3, so f is nonmeasurable on R_C .

If f satisfies (ii), there are $y_1, y_2 \in R$ so that $f(y_1) > f(y_2)$. Pick $b \in]f(y_2), f(y_1)[$. $f^{-1}(]b, \infty[)$ is a ray, and is nonmeasurable in R_C by Corollary 4.3, so f is not measurable on R_C .

In particular, nonconstant polynomials on R are R_C -nonmeasurable by (i) and one-to-one R_u -continuous functions are R_C -nonmeasurable by (ii).

Certain kinds of subsets of R , some of them R_u -open, have been shown to be R_C -nonmeasurable. The next theorem gives some necessary conditions for a subset of R to be R_C -open.

THEOREM 4.4. *If U is an open set in R_c , the diameters of intervals within U' are bounded. If V is a neighborhood of $0 \in R_c$, and $\alpha \in R$, then $\{n_K \alpha\}_{K=1}^\infty \subset V$ where $\{n_K\}_{K=1}^\infty$ is some infinite subset of the positive integers ($\{n_K\}_{K=1}^\infty$ depends on V and α).*

PROOF. Given a nonzero $\alpha \in R$, the identity map from αZ with the discrete topology to R_c is a continuous homomorphism, but not an isomorphism. By (9.1) of [4], given V a neighborhood of $0 \in R_c$, there is $n' = n'(\alpha, V) \in \{1, 2, \dots\}$ such that for every $m \in Z$, $\{m\alpha, (m+1)\alpha, \dots, (n'+m)\alpha\} \cap V$ is nonvoid. Hence there is $n_K \in Z$ such that $2Kn' \leq n_K \leq (2K+1)n'$ and $n_K \alpha \in V$ for $K = 1, 2, 3, \dots$.

Since $\{m\alpha, (m+1)\alpha, \dots, (n'+m)\alpha\} \cap V$ is nonvoid for every $m \in Z$, the intervals contained in $(x+V)'$ for $x \in R$ are bounded. Since V is an arbitrary neighborhood of 0, for U any open set in R_c , the diameter of intervals contained in U' is bounded.

So far, no subsets of R have been shown to be open, closed or even measurable relative to a R_c topology. Even though it is known that R_c is topologically isomorphic to S^r for some r such that $2 \leq 2^r \leq c$, no topological isomorphism between R_c and S^r is known for any r . Finding R_c -measurable subsets of R must be accomplished indirectly. All countable and cocountable sets are R_c -measurable. More interesting R_c -measurable sets may be found using Theorem 3.2. By Theorem 3.2, there is a subgroup H of R which is isomorphic to R , has index c in R and is R_c -dense and Borel. For R_c not topologically isomorphic to S , there is a subgroup J of R which is isomorphic to R , has index c in R and is R_c -closed.

Two unanswered questions brought up by the study of compact topologies for R are:

- (i) If A and A' have nonvoid R_u -interior, must A be R_c -non-measurable?
- (ii) If f is R_u -continuous and R_c -continuous, must f be a constant function?

Besides knowing that no two compact group topologies for R are comparable (since only Hausdorff topologies are considered), no compact group topology for R is comparable to the usual topology of R ((5.29) of [4]).

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