PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 30, No. 3, November 1971

AN EXCEPTIONAL SET FOR INNER FUNCTIONS[1]

RENATE MCLAUGHLIN

ABSTRACT. Suppose f is an inner function that is not a constant and not a finite Blaschke product. Let $E(f)$ denote the set of values inside the unit disk that are not assumed infinitely often by f. We show that $E(f)$ is an F_σ-set.

Let D denote the open unit disk. A function $f(z)$, defined and analytic in D, is called an *inner function* if $|f(z)| \leqq 1$ $(z \in D)$ and if $\lim_{r\to 1} |f(re^{i\theta})| = 1$ for almost all θ $(0 \leqq \theta < 2\pi)$. Throughout this note, we assume that an inner function is neither a constant nor a finite Blaschke product.

With each inner function f, the exceptional set

$$E(f) = \{w: |w| < 1, f \text{ assumes } w \text{ at most finitely often}\}$$

is associated. It is known that $E(f)$ has capacity zero [1, p. 35], and it is also known that for each closed subset S of capacity zero in the open unit disk there exists an inner function that omits every value in S and no other values [1, p. 37], [3].

We prove the following result.

THEOREM. *For each inner function f, the set $E(f)$ is an F_σ-set of capacity zero.*

Suppose f is analytic and nonconstant in D, and let $R(f) = \{w: w = f(z), z \in D\}$. For $n = 1, 2, \cdots$, define

$$E_n(f) = \{w: w \in R(f);\ f \text{ assumes } w \text{ at most } n \text{ times, counting multiplicities}\}.$$

LEMMA. *For each n, $E_n(f)$ is a closed subset of $R(f)$.*

PROOF. We have to show that if $f(z)$ assumes the values $w_1, w_2, \cdots$ $(w_k \in R(f))$ at most n times and if $w_k \to w$ $(w \in R(f))$, then $f(z)$ assumes w at most n times, counting multiplicities.

Suppose $w_1, w_2, \cdots$ belong to $E_n(f)$ and $w_k \to w$ $(w \in R(f))$. Assume there are $n+1$ solutions of the equation $f(z) = w$, say $z_1, \cdots, z_{n+1}$

Received by the editors August 28, 1970.

AMS 1970 *subject classifications.* Primary 30A76; Secondary 30A08.

Key words and phrases. Inner function, values that an inner function assumes at most finitely often, F_σ-set, Rouché's theorem.

[1] This research was supported in part by the National Science Foundation.

($z_k \in D$). Then we can find a simple, closed, rectifiable, positively-oriented curve C in D such that $z_1, \cdots, z_{n+1}$ lie in the interior of C and such that no possible further solutions of $f(z) = w$ lie on C. Set

$$\min_{z \in C} |f(z) - w| = \epsilon.$$

It follows that $\epsilon > 0$. Choose w_m such that $|w - w_m| < \epsilon/2$. There are at most n solutions of the equation $f(z) = w_m$. If necessary, change C to a curve C^* such that none of these solutions lies on C^*. The curve C^* can be chosen in such a way that $z_1, \cdots, z_{n+1}$ are in the interior of C^* and $|f(z) - w| > \epsilon/2$ ($z \in C^*$). Therefore

$$|f(z) - w| > |w - w_m| \qquad (z \in C^*).$$

If we write

$$f(z) - w_m = (f(z) - w) + (w - w_m),$$

all hypotheses of Rouché's theorem are satisfied [2, p. 254], and it follows that $f(z) - w_m$ and $f(z) - w$ have the same number of zeros inside C^*. This is a contradiction. □

If f is an inner function, we define

$$E_0(f) = \{w : |w| < 1, f \text{ omits } w\}.$$

The set $E_0(f)$ is closed, since the range of f is open. The theorem now follows from the fact that $E(f) = \bigcup_{n=0}^{\infty} E_n(f)$.

It is tempting to suspect that the sets

$$E_n^*(f) = \{w : w \in R(f), f \text{ assumes } w \text{ exactly } n \text{ times}\}$$

are closed. This is false. It is not too difficult to think of the Riemann surface of a function that assumes $w = 0$ exactly once and for which $w = 0$ is the limit of both a sequence of points $w_{11}, w_{12}, \cdots$ that are assumed exactly once and a sequence of points $w_{21}, w_{22}, \cdots$ that are assumed exactly twice.

Finally, we remark that there exist inner functions for which $E_0(f) \neq \emptyset$ and each set $E_n(f)$ properly contains $E_{n-1}(f)$.

References

1. E. F. Collingwood and A. J. Lohwater, *The theory of cluster sets*, Cambridge Univ. Press, Cambridge, 1966. MR **38** #325.

2. E. Hille, *Analytic function theory*. Vol. 1, Introduction to Higher Math., Ginn, Boston, Mass., 1959. MR **21** #6415.

3. G. Hössjer and O. Frostman, *Über die Ausnahmestellen eines Blaschkeproduktes*, Kungl. Fysiogr. Sällsk. Lund Förh. **3** (1933), no. 16, 8 pp.

University of Michigan-Flint, Flint, Michigan 48503