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GEODESICS IN METRICAL CONNECTIONS

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ABSTRACT. To each connection on a Riemannian manifold we define a tensor called the Q-tensor. We prove that two metrical connections have the same geodesics if and only if their Q-tensors are equal. We then show that any manifold of dimension greater than two admits many metrical connections having the same geodesics; in particular, the Q-tensor is a strictly weaker invariant than the torsion.

1. Statement of theorems. We shall recall some definitions following [3]. Let M^n be a smooth (C^∞) n-manifold and let $T_x(M)$ be the tangent space at $x \in M$ and $\chi(M)$ the set of all smooth vector fields over M. Let g be a fixed Riemannian metric on M. Let $T^{p,q}(M)$ be the set of tensors of type (p, q). Recall that a (linear) connection on M is a real bilinear operator $D: \chi(M) \times \chi(M) \to \chi(M)$ such that for all C^∞ real valued functions, f, (1) $D_{fX}Y = fD_XY$ and (2) $D_XfY = (Xf)Y + fD_XY$. The torsion of a given connection D, $\mathrm{Tor}_D \in T^{1,2}(M)$, is given by $\mathrm{Tor}(X, Y) = D_XY - D_YX - [X, Y]$ where [,] denotes the Lie bracket. A connection D is called metrical (with respect to g) if:

$$Xg(Y, Z) = g(D_XY, Z) + g(Y, D_XZ)$$

for all $X, Y, Z \in \chi(M)$. A curve α is a geodesic if $D_{d\alpha/dt}d\alpha/dt = 0$. Note that by a geodesic we mean a *parametrized* curve and not just the geometric image of the curve.

The purpose of this note is to answer the following question: When do two metrical connections have the same geodesics? The Levi-Civita Theorem states that a metrical connection (hence its geodesics) is determined uniquely by specifying its torsion. We shall show that the torsion is too strong an invariant for these purposes and that the Q-tensor (to be defined below) is the proper invariant.

Let $\mathrm{Tor}(M) = \{T \in T^{1,2}(M) \mid T(X, Y) = -T(Y, X)$ for all $X, Y \in \chi(M)\}$. For $T \in \mathrm{Tor}(M)$, $\{X_k\}$, $k = 1, \cdots, n$, an orthonormal basis of $T_x(M)$ ($x \in M$), we define $Q^T \in T^{1,2}(M)$ by

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$$\overset{T}{Q}_x(X, Y) = -\frac{1}{2}\sum_{k=1}^{n}\{g(T(X, X_k), Y) + g(T(Y, X_k), X)\}X_k$$

where all tensors on the right-hand side are evaluated at x. A simple computation shows that Q^T is globally well defined. Let $Q(M) = \{Q^T | T\in \mathrm{Tor}(M)\}$ and q: $\mathrm{Tor}(M)\to Q(M)$ be given by $q(T) = Q^T$. If D is a connection on M we define $Q^D = Q^{\mathrm{Tor}_D}$ to be the Q-tensor associated with D.

THEOREM A. *Let M be a Riemannian manifold with metrical connections D and $\overline{D}$, then D and $\overline{D}$ have the same geodesics if and only if $Q^D = Q^{\overline{D}}$.*

Let D be a connection then we may define the torsion transformation $T_Y \in T^{1,1}(M)$ for each Y by $T_Y(X) = \mathrm{Tor}(Y, X)$. Because the Q-tensor is zero for the Riemannian connection we have the following corollary due to E. Cartan [2, p. 59].

COROLLARY. *Let D be a metrical connection on the Riemannian manifold M, then D has the same geodesics as the Riemannian (Levi-Civita) connection if and only if $g(T_Y X, X) = 0$ for all X, $Y\in\chi(M)$.*

From Cartan's viewpoint, the torsion tensor of any connection D splits into several components the sum of two of which is Q^D [2, p. 50–52]. We may interpret this in modern language in terms of an exact sequence of vector bundles over M. For each $\omega\in\Lambda^3(M)$ let $r(\omega)\in T^{1,2}(M)$ be defined by $g(r(\omega)(X, Y), Z) = \omega(X, Y, Z)$. Clearly $r(\omega)\in \mathrm{Tor}(M)$. In classical language, r corresponds to "raise an index." From Theorem A and an easy computation in local coordinates, we have:

THEOREM B. $0\to\Lambda^3(M)\xrightarrow{r}\mathrm{Tor}(M)\xrightarrow{q}Q(M)\to 0$ *is an exact sequence of vector bundles over M.*

COROLLARY. *If D is a metrical connection on M then there is a different metrical connection $\overline{D}$ on M with the same geodesics as D if and only if* $\dim M > 2$.

The corollary follows because metrical connections are determined by their torsion and $\dim \Lambda^3(M) > 0$ if and only if $\dim M > 2$.

2. **Proof of Theorem A.** We fix an orthonormal basis $\{X_i\}$ at the point $x\in M$.

PROPOSITION 1. *Let D be a metrical connection and $\overline{D}$ any connection on the Riemannian manifold M. Let*

$$\overline{D}_X Y - D_X Y = S(X, Y) \quad and \quad S(X_i, X_j) = \sum_{k=1}^{n} S_{ij}^k X_k \quad (near\ x)$$

then $\overline{D}$ *is metrical if and only if*

$$S_{ij}^k = \tfrac{1}{2}\{(\overline{T}_{ij}^k - \overline{T}_{jk}^i + \overline{T}_{ki}^j) - (T_{ij}^k - T_{jk}^i + T_{ki}^j)\} \tag{1}$$

where T_{ij}^k *(resp.* $\overline{T}_{ij}^k$*) are the components of* Tor_D *(resp.* $\mathrm{Tor}_{\overline{D}}$*).*

PROOF.

$$\begin{aligned}\mathrm{Tor}_{\overline{D}}(X, Y) &= D_X Y + S(X, Y) - D_Y X - S(Y, X) - [X, Y] \\ &= \mathrm{Tor}_D(X, Y) + S(X, Y) - S(Y, X);\end{aligned}$$

thus

$$\overline{T}_{ij}^k = T_{ij}^k + S_{ij}^k - S_{ji}^k. \tag{2}$$

By cyclicly permuting (2) we get:

$$\overline{T}_{jk}^i = T_{jk}^i + S_{jk}^i - S_{kj}^i, \tag{3}$$

$$\overline{T}_{ki}^j = T_{ki}^j + S_{ki}^j - S_{ik}^j. \tag{4}$$

(2) − (3) + (4) yields

$$\begin{aligned}&\overline{T}_{ij}^k - \overline{T}_{jk}^i + \overline{T}_{ki}^j \\ &= (T_{ij}^k - T_{jk}^i + T_{ki}^j) - (S_{ji}^k + S_{jk}^i) - (S_{ik}^j - S_{ij}^k) + (S_{kj}^i + S_{ki}^j).\end{aligned} \tag{5}$$

We now write out the condition for $\overline{D}$ to be metrical:

$$\begin{aligned}&g(\overline{D}_X Y, Z) + g(Y, \overline{D}_X Z) \\ &\qquad = g(D_X Y, Z) + g(Y, D_X Z) + g(S(X, Y), Z) + g(Y, S(X, Z)) \\ &\qquad = Xg(Y, Z) + g(S(X, Y), Z) + g(Y, S(X, Z))\end{aligned}$$

thus $\overline{D}$ is metrical if and only if

$$g(S(X, Y), Z) + g(Y, S(X, Z)) = 0 \tag{6}$$

hence if $X = X_j$, $Y = X_i$, $Z = X_k$ then (6) becomes (since the $\{X_j\}$ are orthonormal)

$$S_{ji}^k + S_{jk}^i = 0 \quad \text{for all } i, j, k. \tag{7}$$

Putting (7) into (5)

$$(\overline{T}^k_{ij} - \overline{T}^i_{jk} + \overline{T}^j_{ki}) - (T^k_{ij} - T^i_{jk} + T^j_{ki}) = S^k_{ij} - S^j_{ik}. \tag{8}$$

But (7) also says $0 = S^k_{ij} + S^j_{ik}$ whence adding to (8) gives the result (1). Q.E.D.

We will write Q for Q^D and $\overline{Q}$ for $Q^{\overline{D}}$.

THEOREM 2. *Let M be a Riemannian manifold and let D be a metrical connection. Let $\overline{D}$ be any connection and form $S(X, Y) = \overline{D}_X Y - D_X Y$ then $\overline{D}$ is metrical if and only if*

$$S(X, Y) = \overline{Q}(X, Y) - Q(X, Y) + \frac{\overline{\mathrm{Tor}}\,(X, Y) - \mathrm{Tor}\,(X, Y)}{2}.$$

PROOF. From Proposition 1, in the basis $\{X_j\}$ at the point $x \in M$,

$$2S(X_i, X_j) = \sum_k \{\overline{T}^k_{ij} X_k + (\overline{T}^j_{ki} - \overline{T}^i_{jk}) X_k - (T^k_{ij} X_k + (T^j_{ki} - T^i_{jk}) X_k)\}.$$

$$2S(X_i, X_j) = \overline{\mathrm{Tor}}(X_i, X_j) - \mathrm{Tor}\,(X_i, X_j) + \sum_k (\overline{T}^j_{ki} - \overline{T}^i_{jk}) X_k - \sum_k (T^j_{ki} - T^i_{jk}) X_k. \tag{9}$$

However, if $Q(X_i, X_j) = \sum_{k=1}^n Q^k_{ij} X_k$ then by definition

$$2Q^k_{ij} = -(g(\mathrm{Tor}\,(X_i, X_k), X_j) + g(\mathrm{Tor}\,(X_j, X_k), X_i))$$

so

$$Q^k_{ij} = \tfrac{1}{2}(T^j_{ki} - T^i_{jk}) \tag{10}$$

where the last equality follows from the skew-symmetry of the lower indices of the torsion. Combining this last result with (9) we obtain

$$S(X_i, X_j) = \frac{\overline{\mathrm{Tor}}\,(X_i, X_j) - \mathrm{Tor}\,(X_i, X_j)}{2} + \overline{Q}(X_i, X_j) - Q(X_i, X_j)$$

and so the result follows because S is a tensor. Q.E.D.

We now prove Theorem A. Let

$$\overset{1}{D}_X Y = D_X Y + \frac{\overline{\mathrm{Tor}}\,(X, Y) - \mathrm{Tor}\,(X, Y)}{2}$$

then D^1 and D have the same geodesics [**3**, p. 64], hence D and $\overline{D}$

have the same geodesics if and only if $\bar{D}$ and D^1 have the same geodesics. However, since D and $\bar{D}$ are assumed to be metrical, by Theorem 2 we have

$$\bar{D}_X Y = D_X Y + \bar{Q}(X, Y) - Q(X, Y) + \frac{\overline{\mathrm{Tor}}\,(X, Y) - \mathrm{Tor}\,(X, Y)}{2}\,.$$

Because the last term is skew-symmetric we have [**3**, p. 64] $\bar{D}$ and D^1 have the same geodesics if and only if $Q(X, Y) - \bar{Q}(X, Y)$ is skew-symmetric. However, by its very definition $Q(X, Y)$ is symmetric, hence $Q(X, Y) - \bar{Q}(X, Y)$ is skew-symmetric if and only if $Q(X, Y) - \bar{Q}(X, Y) = 0$. Q.E.D.

The result that 2 metrical connections need not coincide in order to have the same geodesics may also be obtained from two theorems of [**1**, pp. 130–131].

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