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ON THE ABSOLUTE CONTINUITY OF THE LIMIT RANDOM VARIABLE IN THE SUPERCRITICAL GALTON-WATSON BRANCHING PROCESS

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ABSTRACT. Let $\{Z_n : n \geqq 0\}$ be a simple Galton-Watson branching process with offspring distribution $\{p_j\}$ satisying $1 < \sum jp_j < \infty$. It is known that there exist constants C_n such that $W_n \equiv Z_n C_n$ converges with probability one to a nondegenerate limit random variable W. Here we show that this W is always absolutely continuous on $(0, \infty)$.

Let $\{Z_n : n \geqq 0\}$ be a Galton-Watson branching process with offspring probability generating function $f(s) \equiv \sum_{j=0}^{\infty} p_j s^j$. Assume $P(Z_0 = 1) = 1$, $\{p_j\}$ is nondegenerate, $1 < m \equiv \sum jp_j < \infty$ and $p_0 = 0$. Seneta [5] and Heyde [3] have shown that there always exists a sequence of constants $C_n \to \infty$, $C_n^{-1} C_{n+1} \to m$ such that $W_n \equiv Z_n C_n^{-1}$ converges with probability one to a nondegenerate random variable W. Kesten and Stigum [4] had shown earlier that when $\sum p_j j \log j < \infty$, C_n may be taken as m^n and in this case the limit W has an absolutely continuous distribution on $(0, \infty)$. Their main tool was to show that if ϕ is the characteristic function of W then ϕ', the derivative, is integrable. This will fail when $\sum p_j j \log j = \infty$ since in this case $EW = \infty$ and the existence of ϕ' is not guaranteed, let alone its integrability. In this paper we shall present a simple idea to show that W is always absolutely continuous.

If $\phi(it) \equiv E(e^{itW})$ is the characteristic function of W then it is easily seen using the fact $C_n^{-1} C_{n+1} \to m$ that ϕ satisfies the so-called Abel's functional equation.

$$\phi(it) = f(\phi(it/m)). \tag{1}$$

The equation and the nondegeneracy of W ensures that $|\phi(it)| < 1$ for $t \neq 0$. We exploit this to get a rate of decay for $\phi(it)$ as $t \to \infty$. First we recall the following about the rate of convergence of $f_n(x)$ for $|x| < 1$.

LEMMA 1. *If* $p_1 > 0$ *then* $p_1^{-n} f_n(x) \uparrow Q(x) < \infty$ *for* $0 \leqq x < 1$. *If* $p_1 = 0$ *then for any* $\epsilon > 0$, $\lim_{n \to \infty} \epsilon^{-n} f_n(x) = 0$ *for* $0 \leqq x \leqq 1$.

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For a proof see [1].

Let $0<\delta\leqq\infty$ be defined by $p_1=m^{-\delta}$.

LEMMA 2. *If* $p_1>0$ *then* $\sup_t |t|^{\delta}|\phi(it)|<\infty$. *If* $p_1=0$ *then, for any* $\theta>0$, $\sup_t |t|^{\theta}|\phi(it)|<\infty$.

PROOF. By continuity $\beta\equiv\sup_{1\leq|t|\leq m}|\phi(it)|<1$. Iterating (1) we get $\phi(im^n t)=f_n(\phi(it))$ and hence $\sup_{1\leq|t|\leq m}|\phi(im^n t)|\leqq f_n(\beta)$. Now use Lemma 1 to complete the proof. q.e.d.

If $\delta>1$ then Lemma 2 says that ϕ is integrable and so W is absolutely continuous on $(0,\infty)$. In fact, it has a uniformly continuous density function. Assume for the rest of the paper that $\delta\leqq 1$. Let k be the smallest integer such that $k\delta>1$.

LEMMA 3. *For all* $r\geqq k$, *the* r *fold convolution* S_r *of* W *is absolutely continuous on* $(0,\infty)$ *and has a uniformly continuous density.*

PROOF. The characteristic function of S_k is $\phi^k(it)$ and, by Lemma 2, $\sup_t|t|^{k\delta}|\phi^k(it)|<\infty$. But $k\delta>1$ and so ϕ^k and ϕ^r for $r\geqq k$ are integrable. q.e.d.

The following is a key step.

LEMMA 4. *Let* W_j, $j=0,1,2,\cdots$, *be independent random variables with the same distribution as* W. *Assume further that the* W_j *sequence is independent of our Galton-Watson branching process* $\{Z_n\}$. *Then* W_0 *has the same distribution as* $(1/m^n)\sum_{j=1}^{Z_n}W_j$ *for each* n.

PROOF. The characteristic functions of the above two random variables are respectively $\phi(it)$ and $f_n(\phi(it/m^n))$. They are equal for all t as can be seen by iterating (1) n times. q.e.d.

We shall now show that the absolute continuity of S_k for all large k implies the same for W.

LEMMA 5. *Let* E *be a Borel set with Lebesgue measure zero. Then,* $P(W\in E)=0$.

PROOF. From Lemma 4 we see that

$$\begin{aligned} P(W_0\in E) &= \sum_{r=1}^{\infty}P\left(Z_n=r,\frac{1}{m^n}\sum_{j=1}^{Z_n}W_j\in E\right)\\ &= \sum_{r=1}^{\infty}P\left(Z_n=r,\frac{1}{m^n}\sum_{j=1}^{r}W_j\in E\right)\\ &= \sum_{r=1}^{\infty}P(Z_n=r)P\left(\sum_{j=1}^{r}W_j\in m^nE\right)\end{aligned}$$

(by independence of $\{Z_n\}$ and $\{W_j\}$).

But by Lemma 3, for $r \geqq k$, $P(\sum_{j=1}^{r} W_j \in m^n E) = 0$. Thus $P(W_0 \in E) \leqq P(Z_n < k)$ for each n. Clearly, $P(Z_n < k) \to 0$ as $n \to \infty$ for each fixed k. q.e.d.

Lemma 5 does not assert that the density $w(x)$ of W is continuous.[1] If this is the case then the argument in [**2**] yields the conclusion that $w(x) > 0$ for all $x > 0$. One can perhaps then prove a local limit theorem for the $\sum p_j j \log j = \infty$ case in analogy with the results of [**2**].

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[1] Added in proof. S. Dubuc has just shown that $w(x)$ is continuous by using different techniques. He has many more results on this topic. See S. Dubuc, Ann. Inst. Fourier (Grenoble) **21** (1971), 171–251.