PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 30, No. 3, November 1971

A CHARACTERIZATION OF *SH*-SETS

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ABSTRACT. Let G be a locally compact abelian group, and $A(G)$ the Fourier algebra on G. A Helson set in G is called an SH-set if it is also an S-set for the algebra $A(G)$. In this article we prove that a compact subset K of G is an SH-set if and only if there exists a positive constant b such that: For any disjoint closed subsets K_0 and K_1 of K, we can find a function u in $A(G)$ such that $\|u\|<b$, $u=1$ on some neighborhood of K_0, and $u=0$ on some neighborhood of K_1.

It has so far been an open problem whether or not every Helson set in a locally compact abelian group is an S-set. A Helson set is called an SH-set if it is also an S-set. In this paper we give a characterization of SH-sets. Almost all notations, definitions, and terminologies used here are adopted from [1] and [2].

Let G be a locally compact abelian group, $\hat{G}$ its dual, and K any compact subset of G. If K is a quasi-Kronecker set or a K_p-set for some natural number $p \geqq 2$, then there is a positive constant a with the following property:

($\mathcal{K}_a$) For any disjoint closed subsets K_0 and K_1 of K, there is a character γ in $\hat{G}$ such that

$$\inf\{\,|\gamma(x_0)-\gamma(x_1)|\,: x_j \in K_j,\, j=0,1\} \geqq a.$$

Using an analogous argument as in [2, Lemma 7], one can easily prove that every compact set K with property ($\mathcal{K}_a$) satisfies the following condition for some positive constant b:

($\mathcal{K}_b$) For any disjoint closed subsets K_0 and K_1 of K, there is a function u in $A(G)$ such that $\|u\|_{A(G)}<b$ and

$$u(x) = 1 \qquad \text{on some neighborhood of } K_0,$$

$$u(x) = 0 \qquad \text{on some neighborhood of } K_1.$$

We shall verify below that this condition completely characterizes SH-sets, and thus generalize a theorem of N. Th. Varopoulos in [3] and another one of the author in [2, Theorem 11]. Note also that our result is of interest because of the generality, since it is rather trivial for $G=R$ or T (see [3]).

Received by the editors December 7, 1970.

AMS 1970 *subject classifications*. Primary 43A45; Secondary 43A20.

Key words and phrases. Locally compact abelian group, Helson set, S-set, SH-set, quasi-Kronecker set, K_p-set, character, pseudomeasure.

[1] The author wishes to thank the referee for his kind advice.

THEOREM 1. *Let K be any compact subset of G, then K is an SH-set if and only if K satisfies condition* ($\mathcal{K}_b$) *for some positive constant b.*

We need three lemmas.

LEMMA 1. *Let K be a compact subset of G satisfying condition* ($\mathcal{K}_b$) *for some* $b>0$, *and let* $\{K_j\}_1^N$ *be N, pairwise disjoint, closed subsets of K. Then, for any pseudomeasures* P_j *in* $PM(K_j)$ $(j=1, 2, \cdots, N)$, *we have*

$$\sum_{j=1}^{N} \left| \hat{P}_j(\gamma) \right| \leqq 4b \left\| \sum_{j=1}^{N} P_j \right\| \qquad (\gamma \in \hat{G}).$$

PROOF. For given γ in $\hat{G}$, there exists a subset D of the index set $\{1, 2, \cdots, N\}$ such that

$$\sum_{j=1}^{N} \left| \hat{P}_j(\gamma) \right| \leqq 4 \left| \sum_{j\in D} \hat{P}_j(\gamma) \right| \cdot$$

Put $E_0 = \bigcup \{K_j : j \in D\}$ and $E_1 = \bigcup \{K_j : j \in D^c\}$. By hypothesis, there is a function u in $A(G)$ such that $\|u\|_{A(G)} < b$ and $u = 1 - k$ on some neighborhood of E_k $(k=0, 1)$. It follows at once that

$$\left| \sum_{j\in D} \hat{P}_j(\gamma) \right| = \left| \left(u \sum_{j=1}^{N} P_j \right)^{\wedge}(\gamma) \right| \leqq b \left\| \sum_{j=1}^{N} P_j \right\|,$$

which completes the proof.

LEMMA 2. *Let* $\{K_j \subset K\}_1^N$ *and* $\{P_j \in PM(K_j)\}_1^N$ *be as in Lemma* 1, *and let* ϵ *be any positive number. Then, for any character* γ *in* $\hat{G}$ *and any complex numbers* $\{a_j\}_1^N$ *such that*

$$\left| \gamma(x) - a_j \right| < \epsilon \qquad (x \in K_j), \ \left| a_j \right| = 1 \quad (j = 1, 2, \cdots, N),$$

we have

$$\left| \sum_{j=1}^{N} \hat{P}_j(\gamma) - \sum_{j=1}^{N} a_j \hat{P}_j(0) \right| \leqq b_1 \cdot \left\| \sum_{j=1}^{N} P_j \right\| \cdot \epsilon,$$

where b_1 *is a constant depending only on b.*

PROOF. For given $\epsilon > 0$, we can find a function g in $A(T)$ such that

$$g(e^{it}) = 1 - e^{it} \quad (\left| 1 - e^{it} \right| < \epsilon) \quad \text{and} \quad \|g\|_{A(T)} < M \cdot \epsilon,$$

where M is an absolute constant (cf. [4, pp. 80–81]). Then, if γ and $\{a_j\}_1^N$ are as above, it is easy to see that

$$\gamma(x) - a_j = \sum_{n=-\infty}^{\infty} \hat{g}(n) \cdot \gamma(-(n-1)x) \cdot a_j^n$$

on some neighborhood of K_j $(j=1, 2, \cdots, N)$. It follows that

$$\left| \sum_{j=1}^{N} \hat{P}_j(\gamma) - \sum_{j=1}^{N} a_j \hat{P}_j(0) \right| = \left| \sum_{n=-\infty}^{\infty} \hat{g}(n) \sum_{j=1}^{N} a_j^n \hat{P}_j(-(n-1)\gamma) \right|$$

$$\leqq \sum_{n=-\infty}^{\infty} | \hat{g}(n) | \cdot \sup_{\lambda \in \hat{G}} \sum_{j=1}^{N} | \hat{P}_j(\lambda) |$$

$$= \|g\|_{A(T)} \cdot \sup_{\lambda \in \hat{G}} \sum_{j=1}^{N} | \hat{P}_j(\lambda) | .$$

Therefore, the required conclusion immediately follows from Lemma 1, which completes the proof.

LEMMA 3. *Under the hypotheses of Lemma* 1, *we have*

$$\sum_{j=1}^{N} \|P_j\| \leqq b_2 \left\| \sum_{j=1}^{N} P_j \right\| ,$$

where b_2 is a constant depending only on b.

PROOF. Let $\{\gamma_j\}_1^N$ be any characters in $\hat{G}$, and ϵ any positive number. Then, by taking a sufficiently large natural number n, we can find pairwise disjoint closed subsets $\{K_{j,k}\}_{k=1}^n$ of K_j $(j=1, 2, \cdots, N)$ so that: For any choice of k $(=1, 2, \cdots, n)$, there correspond pairwise disjoint closed subsets $\{K_{j,k,l}\}_{l=1}^n$ of K_j such that

$$K_j \backslash K_{j,k} \subset \bigcup_{l=1}^{n} K_{j,k,l} \tag{1}$$

and

$$x, y \in K_{j,k,l} \Rightarrow | \gamma_j(x) - \gamma_j(y) | < \epsilon \quad (l = 1, 2, \cdots, n). \tag{2}$$

In order to construct such sets $K_{j,k}$, it suffices to modify the proof of Lemma 10 in [2].

Let now $\{U_{j,k}\}$ be pairwise disjoint open neighborhoods of $\{K_{j,k}\}$, and choose a function u in $A(G)$ such that $\|u\| < b$ and

$$u = 1 \quad \text{on some neighborhood of } \bigcup_{j=1}^{N} \bigcup_{k=1}^{n} K_{j,k},$$

$$u = 0 \quad \text{on some neighborhood of } K \Big\backslash \left(\bigcup_{j=1}^{N} \bigcup_{k=1}^{n} U_{j,k} \right).$$

Then, for each $j=1, 2, \cdots, N$, we have a decomposition of uP_j of the form

$$uP_j = \sum_{k=1}^{n} P_{j,k} \qquad \text{where supp } P_{j,k} \subset U_{j,k} \cap K_j.$$

It follows from Lemma 1 that we have

$$| \widehat{P}_{j,k}(\gamma_j) | \leqq 4b\|uP_j\|/n \leqq 4b^2\|P_j\|/n$$

for some $k=k_j$; without loss of generality, we may assume $k_j=1$. Therefore we have

$$\sum_{j=1}^{N} | P_{j,1}(\gamma_j) | \leqq 4b^2 \sum_{j=1}^{N} \|P_j\|/n. \tag{3}$$

Choose then any function v in $A(G)$ such that $\|v\| < b$ and

$$v = 1 \quad \text{on some neighborhood of } \bigcup_{j=1}^{N} (K_{j,1} \cup \text{supp } P_{j,1}),$$

$$v = 0 \quad \text{on some neighborhood of } \bigcup_{j=1}^{N} \bigcup_{k=2}^{n} \text{supp } P_{j,k}.$$

It follows at once that

$$vu \sum_{j=1}^{N} P_j = \sum_{j=1}^{N} P_{j,1} \tag{4}$$

and

$$\text{supp}\left(w \sum_{j=1}^{N} P_j\right) \subset \bigcup_{j=1}^{N} K_j \backslash K_{j,1}, \tag{5}$$

where $w=1-vu$. By (1) and (5), there are $N \times n$ pseudomeasures $Q_{j,l}$ such that

$$wP_j = \sum_{l=1}^{n} Q_{j,l} \quad \text{and} \quad \text{supp } Q_{j,l} \subset K_{j,1,l} \tag{6}$$

for all $j=1, 2, \cdots, N$. By (2), there are $N \times n$ complex numbers $a_{j,l}$ with $|a_{j,l}|=1$ such that

$$| \gamma_j(x) - a_{j,l} | < \epsilon \qquad (x \in K_{j,1,l};\ 1 \leqq j \leqq N;\ 1 \leqq l \leqq n). \tag{7}$$

It then follows from (4) and (6) that

$$\begin{aligned} | \hat{P}_j(\gamma_j) | &\leqq | \widehat{P}_{j,1}(\gamma_j) | + | \widehat{wP}_j(\gamma_j) | \\ &\leqq | \widehat{P}_{j,1}(\gamma_j) | + \left| \widehat{wP}_j(\gamma_j) - \sum_{l=1}^{n} a_{j,l}\widehat{Q}_{j,l}(0) \right| + \sum_{l=1}^{n} | \widehat{Q}_{j,l}(0) |, \end{aligned}$$

which combined with (7) and Lemma 2 yields

$$\begin{aligned}|\hat{P}_j(\gamma_j)| &\leqq |\widehat{P_{j,1}}(\gamma_j)| + b_1\|wP_j\|_\epsilon + \sum_{l=1}^{n} |\widehat{Q_{j,l}}(0)| \\ &\leqq |\widehat{P_{j,1}}(\gamma_j)| + b_1(1+b^2)\|P_j\|_\epsilon + \sum_{l=1}^{n} |\widehat{Q_{j,l}}(0)|.\end{aligned}$$

Summing up these inequalities for $j=1, 2, \cdots, N$, we have by (3),

$$\sum_{j=1}^{N} |\hat{P}_j(\gamma_j)| \leqq 4b^2 \sum_{j=1}^{N} \|P_j\|/n + b_1(1+b^2) \sum_{j=1}^{N} \|P_j\|_\epsilon + \sum_{j=1}^{N}\sum_{l=1}^{n} |\widehat{Q_{j,l}}(0)|.$$

But we have also, by Lemma 1 and (6),

$$\sum_{j=1}^{N}\sum_{l=1}^{n} |\widehat{Q_{j,l}}(0)| \leqq 4b\left\| \sum_{j=1}^{N}\sum_{l=1}^{n} Q_{j,l}\right\| \leqq 4b(1+b^2)\left\| \sum_{j=1}^{N} P_j\right\|.$$

Thus, letting $n \to +\infty$ and $\epsilon \to +0$, we obtain

$$\sum_{j=1}^{N} |\hat{P}_j(\gamma_j)| \leqq 4b(1+b^2)\left\| \sum_{j=1}^{N} P_j\right\|.$$

Finally, since γ_j are arbitrary characters, we have

$$\sum_{j=1}^{N} \|P_j\| \leqq b_2\left\| \sum_{j=1}^{N} P_j\right\| \quad \text{with } b_2 = 4b(1+b^2),$$

which establishes our lemma.

Proof of Theorem 1. Suppose that K is an SH-subset of G, and let K_0 and K_1 be any disjoint closed subsets of K. Take then any function u_0 in $A(G)$ such that $u_0 = 1-j$ on some neighborhood of K_j ($j=0, 1$). Since K is a Helson set, there is a function f in $A(G)$ such that $f=1-j$ on K_j ($j=0, 1$), and $\|f\| < b$, where b is a constant depending only on K. Since $K_0 \cup K_1$ is an S-set, we can choose a function g in $I_0(K_0\cup K_1)$ so that $\|u_0 - f - g\| < b - \|f\|$. Therefore the function $u = u_0 - g$ satisfies the required condition.

Conversely, suppose that K is a compact subset of G satisfying condition $(\mathcal{H}_b)$ for some positive number b. Taking any pseudo-measure P in $PM(K)$, we must prove that P is a measure on K. The needed argument is almost all identical with that in the proof of Theorem 11 in [2]. Let $\{\gamma_j\}_1^N$ be any N characters in $\hat{G}$, and ϵ any positive number. Using Lemma 3, we can find a pseudomeasure

$P^{(1)}$ in $PM(K)$ such that $\|P-P^{(1)}\|<\epsilon/N$ and supp $P^{(1)}\subset\bigcup_k K'_k$, where $\{K'_k\}_k$ are finitely many, pairwise disjoint, closed subsets of K such that

$$x, y \in K'_k \Rightarrow |\gamma_1(x) - \gamma_1(y)| < \epsilon.$$

In fact, in the proof of Lemma 3, let $N=1$, $K_1=K$, $P_1=P$, and, for a given natural number n, construct n pseudomeasures $\{P_k=P_{1,k}\}_{k=1}^n$ as there. By Lemma 3, we then have

$$\|P_k\| \leqq b_2\cdot\|uP\|/n \leqq b_2\cdot b\cdot\|P\|/n$$

for some k. Therefore it suffices to set $P^{(1)}=P-P_k$ for a sufficiently large n and some k $(=1, 2, \cdots, n)$.

Repeating the same arguments for supp $P^{(1)}$, $P^{(1)}$, and γ_2, and so on, we obtain a pseudomeasure $Q=P^{(N)}$ in $PM(K)$ such that $\|P-Q\|<\epsilon$ and supp $Q\subset\bigcup_l K_l$, where $\{K_l\}_l$ are infinitely many, pairwise disjoint, closed subsets of K such that

$$x, y \in K_l \Rightarrow |\gamma_j(x) - \gamma_j(y)| < \epsilon \qquad (\forall l, j).$$

Let $\{Q_l\}_l$ be the pseudomeasures such that $Q=\sum_l Q_l$ and supp $Q_l\subset K_l$, and let $\{x_l\in K_l\}_l$ be any choice of points; we define a measure $\mu\in M(K)$ by $\mu=\sum_l \hat{Q}_l(0)\delta_l$, where δ_l is the unit mass at x_l. We then have

$$\|\mu\|_{M(K)} = \sum_l |\hat{Q}_l(0)| \leqq 4b\|Q\| \leqq 4b(1+\epsilon)\|P\|$$

by Lemma 1, and

$$|\hat{P}(\gamma_j) - \hat{\mu}(\gamma_j)| \leqq |\hat{P}(\gamma_j) - \hat{Q}(\gamma_j)| + \left|\hat{Q}(\gamma_j) - \sum_l \gamma_j(x_l)\hat{Q}_l(0)\right|$$
$$< \epsilon + b_1\|Q\|\epsilon \leqq \epsilon\{1 + b_1(1+\epsilon)\|P\|\} \qquad (j = 1, 2, \cdots, N)$$

by Lemma 2. Therefore we can easily prove that P is a measure on K.

This completes the proof of Theorem 1.

THEOREM 2. *Let K be any totally disconnected compact subset of G, then K is an SH-set if and only if K satisfies the following condition for some positive constant c:*

($\mathcal{K}'_c$) *For any pairwise disjoint closed subsets $\{K_j\}_1^N$ of K, and any pseudomeasures $\{P_j\}_1^N$ with* supp $P_j\subset K_j$, *we have*

$$\sum_{j=1}^N |\hat{P}_j(0)| \leqq c\left\|\sum_{j=1}^N P_j\right\|.$$

PROOF. The proof is essentially contained in that of Theorem 1 (see also [**3**]). We omit the details.

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