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SUBPARACOMPACTNESS AND G_δ-DIAGONALS IN ARHANGEL'SKIĬ'S CLASS MOBI

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ABSTRACT. In this note it is shown that a topological space in Arhangel'skiĭ's class MOBI need not be subparacompact nor have a G_δ-diagonal.

1. **Introduction.** In [1] Arhangel'skiĭ introduced the *class of topological spaces* MOBI and posed Group of Problems 5.7 concerning the topological properties possessed by spaces in MOBI. Most of these questions were answered negatively in [3]. The purpose of this paper is to show that spaces in MOBI need not be subparacompact nor need they have a G_δ-diagonal.

The class of *F_σ-screenable spaces* was introduced by McAuley [12] and the class of *σ-paracompact spaces* was introduced by Arhangel'skiĭ [1]. In [6], Burke showed that the two classes were equivalent and called the common concept *subparacompactness*.

The notion of a G_δ-diagonal is not new but has recently been used by Borges [5] and Lutzer [10].

Recall that an *open-compact map* is a map (=continuous function) such that the images of open sets are open and the inverse images of points are compact. In [8], S. Hanai has shown that every Hausdorff open compact image of a metric space is a Hausdorff, metacompact developable space. In [3] it is shown that any Hausdorff, metacompact developable space is in MOBI. R. H. Bing [4] has shown that developable spaces are F_σ-screenable (hence subparacompact). It is well known that the product of two developable spaces is again developable and that developable spaces have closed sets G_δ [13]. Thus Hausdorff developable spaces have G_δ-diagonals. It follows that every Hausdorff open-compact image of a metric space is subparacompact and has a G_δ-diagonal.

2. **Definitions and preliminaries.** In [3] the following definition is shown to be equivalent to Arhangel'skiĭ's original definition of the class MOBI.

(2.1) DEFINITION. A topological space X is in *the class* MOBI if there is a metric space M and a finite set $\{\phi_1, \cdots, \phi_n\}$ of open-compact maps such that $(\phi_n \circ \cdots \circ \phi_1)(M) = X$.

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It is clear that any open-compact image of a space in MOBI is again in MOBI.

In order to see some types of nonmetrizable spaces in MOBI the following definition is needed.

(2.2) DEFINITION. A sequence $G_1, G_2, \cdots$ of collections of open subsets of a topological space X is called a quasi-development for X provided that for each point p of X and each open set R containing p there is a natural number n such that p belongs to some element of G_n and each element of G_n that contains p lies in R. If, in addition, each G_n is a cover of X, then $G_1, G_2, \cdots$ is a development. A space having a (quasi-) development is said to be (quasi-) developable. Developable spaces are studied extensively in [13] and quasi-developable spaces are studied in [2].

The first formal definition of *countable subparacompactness* was given by Hodel [9] although such spaces were studied by Mansfield [11].

(2.3) DEFINITION. A topological space is said to be (*countably*) *subparacompact* if each (countable) open cover has a σ-discrete closed refinement.

Notice that a paracompact space is subparacompact and a subparacompact space is countably subparacompact.

(2.4) DEFINITION. A topological space X has a *G_δ-diagonal* if $\{(x, x): x \in X\}$ is a G_δ-subset of $X \times X$.

Let all undefined terms and notation be as in [7] except let R denote the set of real numbers and E^1 denote R with the usual Euclidean topology. In particular let Z^+ denote the set of natural numbers. All spaces considered are at least Hausdorff.

3. Subparacompactness and G_δ-diagonals in MOBI.

The next theorem shows the existence of a space in MOBI that is not countably subparacompact and, hence, not subparacompact.

(3.1) THEOREM. *There exists a Hausdorff space Y in* MOBI *that is not countably subparacompact.*

PROOF. To obtain Y, a metacompact developable space X and an open-compact map ϕ are exhibited such that $\phi(X) = Y$.

Let $X = R \times \{0\} \cup (R - Q) \times \{1/n : n \in Z^+\}$. Let $S_n(x, y)$ be defined for each $n \in Z^+$ and $(x, y) \in X$ by

(i) $S_n(x, y) = \{(x, y)\}$ if $y > 0$,

(ii) $S_n(x, y) = \{(x, 0)\} \cup \{(x, z) \in X : z \leq 1/n\}$ if $x \in R - Q$ and $y = 0$, and

(iii) $S_n(x, y) = \{(a, b) \in X : |b| < 1/n \text{ and } \mathrm{Tan}^{-1}|(x - a)/b| < 1/n\} \cup \{(x, 0)\}$ if $x \in Q$ and $y = 0$.

Topologize X by letting $\mathcal{S}=\{S_n(x, y):n\in Z^+, (x, y)\in X\}$ be a subbase for X. If $G_n=\{S_n(x, y):(x, y)\in X\}$, then $\mathcal{G}=\{G_n:n\in Z^+\}$ is a development for X. It may be shown that X is a metacompact completely regular space and, thus, is in MOBI. It is well known that developable spaces have closed sets G_δ [13] and Hodel [9] has shown that any space that has closed sets G_δ is countably subparacompact. Thus X is countably subparacompact.

Let $Y=R$ and topologize Y by letting sets of the following types form a subbase for Y:

(i) $\{x\}$ if $x\in R-Q$, and

(ii) if $x\in Q$, $\{y\in R-Q:|x-y|<1/n\}\cup\{x\}$ for some $n\in Z^+$. It is easily seen that Y is a nonregular quasi-developable space.

Let ϕ be a map from X onto Y defined by $\phi(x, y)=x$. That ϕ is an open-compact map is readily verified and, thus, Y is in MOBI.

To see that Y is not countably subparacompact a countable open cover with no σ-discrete closed refinement must be exhibited. To this end if $r\in Q$, then $U(r)=\{r\}\cup(R-Q)$ is an open set and $\mathcal{U}=\{U(r):r\in Q\}$ is a countable open covering of X. Suppose that $\mathcal{B}=\bigcup\{B_i:i\in Z^+\}$ is a σ-discrete closed refinement of $\mathcal{U}$. Notice that if $b\in\mathcal{B}$, then $b\cap Q$ is at most one point. It follows that $\mathrm{cl}(b, E^1)$ (=closure of b with respect to E^1) is nowhere dense in E^1 for if not, then there exist rationals r and s in $\mathrm{cl}(b, E^1)$ which implies r and s are in b, a contradiction. Since each B_i is a discrete collection a similar argument shows that $A_i=\bigcup\{b\in B_i:b\cap Q=\varnothing\}$ is nowhere dense in E^1 for each $i\in Z^+$. Again since each B_i is a discrete collection its members are disjoint. If $r\in Q$, let $C(r, i)$ be the element of B_i (if it exists) that contains r. By an argument used above $\mathrm{cl}(C(r, i), E^1)$ is nowhere dense in E^1. Thus

$$E^1=\bigcup\{A_i:i\in Z^+\}\cup(\bigcup\{C(r, i):i\in Z^+,r\in Q\})$$

which violates E^1 being of the second category. Thus $\mathcal{U}$ has no σ-discrete closed refinement and Y is not countably subparacompact.

(3.2) COROLLARY. *An open-compact map need not preserve countable subparacompactness.*

It is of more than passing interest to note that the space Y in (3.1) is quasi-developable since in [3] it is asked if each space in MOBI is quasi-developable. The space Y also shows that quasi-developable spaces need not be countably subparacompact although developable spaces are.

(3.3) THEOREM. *There exists a Hausdorff space Y in* MOBI *that does not have a G_δ-diagonal.*

PROOF. Let $Y=\{(x, 0):x\in Q\}\cup\{(x, y):x\in R-Q$ and $y=1/n$ or $y=1-1/n, n\in Z^+, n\geqq 2\}$. The lexicographic ordering on Y induces a topology in Y such that Y is a linearly ordered topological space. It is easily seen that $\{(x, y)\in Y:x\in Q\}$ is a closed subset of Y but not a G_δ-subset of Y. Thus Y is nonmetrizable. In [10] it is shown that any linearly ordered topological space with a G_δ-diagonal is metrizable. Thus Y does not have a G_δ-diagonal. Notice that Y is quasi-developable.

Let $X=\{(x, 0, 0):x\in Q\}\cup\{(x, y, z):x\in R-Q, y=1/n$ or $y=1-1/n, n\in Z^+, n\geqq 2, z\in Z^+\cup\{0\}\}$. Let $S_n(x, y, z)$ be defined for each $n\in Z^+$ and $(x, y, z)\in X$ by

(i) $S_n(x, y, z)=\{(a, b, c)\in X:|x-a|<1/n, b=0, c=0$ or $|x-a|<1/n, 0<b<1, c\geqq n\}$ if $x\in Q$,

(ii) $S_n(x, y, z)=\{(x, y, c)\in X:c=0$ or $c\geqq n\}$ if $x\in R-Q$ and $z=0$, and

(iii) $S_n(x, y, z)=\{(x, y, z)\}$ if $x\in R-Q$ and $z>0$. Topologize X by letting $\{S_n(x, y, z):n\in Z^+, (x, y, z)\in X\}$ be a subbase. Let $G_n=\{S_n(x, y, z):(x, y, z)\in X\}$. It may be shown that $\{G_n:n\in Z^+\}$ is a development for X and that X is a nonregular, metacompact, developable space. Thus X has a G_δ-diagonal and is in MOBI.

The map ϕ defined by $\phi(x, y, z)=(x, y)$ is readily verified to be an open-compact mapping of X onto Y. Thus Y is in MOBI.

(3.4) COROLLARY. *An open-compact map does not necessarily preserve the property of having a G_δ-diagonal.*

Notice that the examples used in (3.1) and (3.3) are Hausdorff spaces. It is an open question if spaces with stronger separation axioms can be found that are in MOBI and that are not subparacompact and do not have G_δ-diagonals.

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