AN EXTREMAL PROPERTY OF SIMPLICES

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ABSTRACT. It is shown that the class of n -dimensional simplices is an extremal for the problem of maximizing the volume of the minimum circumscribing cylinder to an n -dimensional convex body of unit volume.

1. It is a simple consequence of a result of Macbeath (Lemma 6 of [1]) that every compact convex body K in E^n is contained in a cylinder whose volume is no greater than n times that of K . By a cylinder, we shall mean any affine equivalent of the cartesian product of an $(n-1)$ -dimensional convex set and a line-segment.

The purpose of this note is to show that the class of n -dimensional simplices is an extremal for the problem of maximizing the volume of the circumscribing cylinder to a convex body. This will follow from the result quoted above and the following

THEOREM. *Every circumscribing cylinder to an n -dimensional simplex has volume at least n times that of the simplex.*

We note that a standard approximation argument using the Blaschke selection theorem (see, for example, [4]) shows that for any compact convex body K in E^n , the value

$$\inf_{C \supset K} \mu_n(C), \quad C \text{ a cylinder,}$$

where $\mu_n(\cdot)$ denotes n -dimensional measure, is attained for some cylinder $C' \supset K$.

We introduce some notation. Let $\mathbf{x} = (\xi_1, \dots, \xi_n)$ be a general point of E^n . We shall denote the hyperplane $\xi_n = 0$ by P_0 , and the parallel plane $\xi_n = 1$ by P_1 . If $A \subset E^n$ is a compact set, we use $\text{conv } A$ to denote its convex hull, and if $\mathbf{a}$ is a vector in E^n which is not parallel to P_0 , we shall use $\pi(\mathbf{a}, A)$ to denote the shadow of A on P_0 in the direction $\mathbf{a}$, that is

$$\pi(\mathbf{a}, A) = \{ \mathbf{x} + t\mathbf{a} \mid \mathbf{x} \in A, -\infty < t < \infty \} \cap P_0.$$

If S is an n -dimensional simplex, we define $V(S)$ to be its vertex set and $V_i(S)$ to be $V(S) \cap P_i$, $i = 0, 1$.

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2. In this section we shall prove a number of lemmas that we shall use in proving the Theorem. Before stating these, however, we remark that since the problem is invariant under affine transformations, we need to concern ourselves only with those simplices S for which P_0 and P_1 are tac-planes (we shall denote the family of such simplices by $\mathcal{S}$) and with those cylinders whose bases are shadows on to P_0 of S in the direction $\mathbf{a}$ of their generators. Accordingly, the problem becomes that of minimizing

$$f(\mathbf{a}, S) = \mu_{n-1}[\pi(\mathbf{a}, S)] / \mu_n(S),$$

for $S \in \mathcal{S}$ and $\mathbf{a}$ nonparallel to P_0 . The lemmas are

LEMMA 1. *Let $S \in \mathcal{S}$ and let $\mathbf{a}$ be a vector, nonparallel to P_0 . Then there exists a simplex $S^* \in \mathcal{S}^*$, the family of n -dimensional simplices whose vertex sets are contained in $P_0 \cup P_1$, for which $f(\mathbf{a}, S^*) \leq f(\mathbf{a}, S)$.*

LEMMA 2. *Let $S \in \mathcal{S}^*$ and suppose that $V_0(S) \cap \pi(\mathbf{a}, V_1(S)) \neq \emptyset$. Then $f(\mathbf{a}, S) = n$.*

LEMMA 3. *Let $S \in \mathcal{S}^*$ and let $\mathbf{a}$ be a vector nonparallel to P_0 . Suppose, further, that $\text{conv}(V_0(S)) \cap \text{conv}[\pi(\mathbf{a}, V_1(S))] \neq \emptyset$. Then $f(\mathbf{a}, S) \geq n$.*

LEMMA 4. *Let X_1, X_2 be subsets of E^n . Then $\text{conv } X_1 + \text{conv } X_2 = \text{conv}(X_1 + X_2)$.*

PROOF OF LEMMA 1. Let $V(S) = \{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n\}$. If $S \in \mathcal{S}^*$, there is nothing to prove. Suppose, then, that one of the vertices, say $\mathbf{x}_0$, lies strictly between P_0 and P_1 . Then there exist numbers t_0, t_1 of opposite sign, for which

$$\mathbf{x}_0 + t_i \mathbf{a} \in P_i, \quad i = 0, 1.$$

Let $\mathbf{b}$ be a unit vector orthonormal to the hyperplane containing $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$, and let T denote $\text{conv}(\bigcup_{i=0}^n \mathbf{x}_i)$. Then for any real number t , we have

$$\begin{aligned} \mu_n[\text{conv}\{\mathbf{x}_0 + \mathbf{a}, \mathbf{x}_1, \dots, \mathbf{x}_n\}] &= \frac{1}{n} |(\mathbf{x}_0 + t\mathbf{a}) \cdot \mathbf{b}| \mu_{n-1}(T) \\ &= \left| \mu_n(S) + \frac{t}{n} (\mathbf{a} \cdot \mathbf{b}) \right| \mu_{n-1}(T). \end{aligned}$$

It follows that one of the two simplices

$$S_i = \text{conv}\{\mathbf{x}_0 + t_i \mathbf{a}, \mathbf{x}_1, \dots, \mathbf{x}_n\}, \quad i = 0, 1,$$

satisfies the inequalities

$$\mu_n(S_i) \geq \mu_n(S), \quad \mu_{n-1}[\pi(\mathbf{a}, S_i)] = \mu_{n-1}[\pi(\mathbf{a}, S)],$$

and

$$\text{card}(V_0(S_i) \cup V_1(S_i)) > \text{card}(V_0(S) \cup V_1(S)).$$

Repeating this argument as many times as necessary, we obtain a simplex with the desired properties.

PROOF OF LEMMA 2. Without loss of generality, suppose that $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_h \in P_0, \mathbf{x}_{h+1}, \mathbf{x}_{h+2}, \dots, \mathbf{x}_n \in P_1$, where $V(S) = \{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n\}$, and $0 \leq h \leq n-1$, and that $\pi(\mathbf{a}, \mathbf{x}_n) = \mathbf{x}_0$.

Then we can find an affine transformation of E^n that takes $\mathbf{x}_0$ into $\mathbf{0}$, $\mathbf{x}_i$ into $\mathbf{e}_i$, $i=1, \dots, h$, $\mathbf{x}_i$ into $\mathbf{e}_i + \mathbf{e}_n$, $i=h+1, \dots, n-1$, and $\mathbf{x}_n$ and $\mathbf{a}$ into $\mathbf{e}_n$, where $\mathbf{e}_1 = (1, 0, \dots, 0, 0)$, $\mathbf{e}_2 = (0, 1, \dots, 0, 0)$, $\dots$, $\mathbf{e}_n = (0, 0, \dots, 0, 1)$.

Then under this transformation it is immediate that $\mu_n(S_n) = 1/n!$, $\mu_{n-1}[\pi(\mathbf{a}, S_n)] = 1/(n-1)!$, and the result follows.

PROOF OF LEMMA 3. Assume, as in Lemma 2, that $\bigcup_{i=0}^h \mathbf{x}_i \subset P_0$, $\bigcup_{i=h+1}^n \mathbf{x}_i \subset P_1$, with $0 \leq h \leq n-1$, where $V(S) = \{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n\}$. Suppose that $\mathbf{c} \in \text{conv } V_0(S) \cap \text{conv}[\pi(\mathbf{a}, V_1(S))]$ and let $\mathbf{d} \in \text{conv } V_1(S)$ such that $\pi(\mathbf{a}, \mathbf{b}) = \mathbf{c}$. Define S_i to be $\text{conv}[\mathbf{d} \cup V(S) \sim \mathbf{x}_i]$, $i=h+1, \dots, n$, and A_i to be $\pi(\mathbf{a}, S_i)$. Then the S_i are (possibly degenerate) simplices and the A_i are their shadows onto P_0 in the direction $\mathbf{a}$. We assert that

- (a) $\bigcup_{i=h+1}^n S_i = S$;
- (b) $\bigcup_{i=h+1}^n A_i = \pi(\mathbf{a}, S)$;
- (c) $\mu_n(S_i \cap S_j) = 0$ for $i \neq j$;
- (d) $\mu_{n-1}(A_i \cap A_j) = 0$ for $i \neq j$.

We omit the proof which is an elementary application of vector algebra. From (a)–(d) we deduce the existence of an n -dimensional simplex $S_m \in \mathcal{S}^*$ where $h+1 \leq m \leq n$ such that $f(\mathbf{a}, S_m) \leq f(\mathbf{a}, S)$, and $\text{conv } V_0(S_m) \cap \pi[\mathbf{a}, V_1(S_m)] \neq \emptyset$.

Now we repeat the above argument with S_m in place of S and with the roles of P_0 and P_1 interchanged to obtain a simplex $\bar{S} \in \mathcal{S}^*$ which satisfies the conditions of Lemma 2 and for which $f(\mathbf{a}, \bar{S}) \leq f(\mathbf{a}, S)$. Lemma 3 now follows.

PROOF OF LEMMA 4. Let $\mathbf{x} \in \text{conv}(X_1 + X_2)$. Then $\mathbf{x} = \lambda(\mathbf{a}_1 + \mathbf{a}_2) + (1-\lambda)(\mathbf{b}_1 + \mathbf{b}_2)$, where $0 \leq \lambda \leq 1$, $\mathbf{a}_i \in X_i$, $\mathbf{b}_i \in X_i$, $i=1, 2$, and so $\mathbf{x} = (\lambda\mathbf{a}_1 + (1-\lambda)\mathbf{b}_1) + (\lambda\mathbf{a}_2 + (1-\lambda)\mathbf{b}_2) \in \text{conv } X_1 + \text{conv } X_2$. Thus $\text{conv}(X_1 + X_2) \subset \text{conv } X_1 + \text{conv } X_2$. On the other hand suppose $\mathbf{y} \in \text{conv } X_1 + \text{conv } X_2$. Then $\mathbf{y} = \lambda_1\mathbf{a}_1 + (1-\lambda_1)\mathbf{b}_1 + \lambda_2\mathbf{a}_2 + (1-\lambda_2)\mathbf{b}_2$, where $0 \leq \lambda_i \leq 1$, $\mathbf{a}_i \in X_i$, $\mathbf{b}_i \in X_i$, $i=1, 2$. Without loss of generality, assume that $\lambda_1 \leq \lambda_2$. Then $\mathbf{y} = \lambda_1(\mathbf{a}_1 + \mathbf{a}_2) + (\lambda_2 - \lambda_1)(\mathbf{b}_1 + \mathbf{a}_2)$

$+(1-\lambda_2)(b_1+b_2) \in \text{conv}(X_1+X_2)$. Thus $\text{conv}(X_1+X_2) \subset \text{conv } X_1 + \text{conv } X_2$ and the lemma is proved.

3. We are now ready to prove the theorem. After Lemma 1, we need only consider simplices in $\mathcal{S}^*$. Let $S \in \mathcal{S}^*$, $\mathbf{a}$ be any vector non-parallel to P_0 . Let $V(S) = \{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n\}$ and assume that

$$\bigcup_{i=0}^h \mathbf{x}_i \subset P_0, \quad \bigcup_{i=h+1}^n \mathbf{x}_i \subset P_1, \quad \text{where } 0 \leq h \leq n-1.$$

Let $\mathbf{x}'_i$ be $\pi(\mathbf{e}_n, \mathbf{x}_i)$, $i=0, 1, \dots, n$. Define D to be the set

$$\{\mathbf{u} \mid \mathbf{u} \in P_0 \text{ and } \text{conv } V_0(S) \cap \text{conv}[\pi(\mathbf{e}_n + \mathbf{u}, V_1(S))] \neq \emptyset\}.$$

We shall show that

(1) D is the $(n-1)$ -dimensional convex hull H of the set of vectors

$$\mathbf{y}_{ij} = \mathbf{x}'_i - \mathbf{x}'_j, \quad i = 0, 1, \dots, h; \quad j = h+1, \dots, n.$$

In fact, we can show that D is a convex polytope with $(h+1)(n-h)$ vertices, but this fact will not be needed and so the proof is omitted. Clearly $D \supset H$. Let $\mathbf{x} \in D$. Then there is a vector $\mathbf{y} \in \text{conv } V_0(S) \cap \text{conv}[\pi(\mathbf{e}_n + \mathbf{x}, V_1(S))]$, and so $\mathbf{y} = \mathbf{c} + \mathbf{d} + \mathbf{x}$, where $\mathbf{c} \in \text{conv } V_0(S)$, and $\mathbf{d} \in \text{conv } \pi(\mathbf{e}_n, V_1(S))$, and so $\mathbf{x} \in \text{conv } V_0(S) - \text{conv } \pi(\mathbf{e}_n, V_1(S))$. By Lemma 4, $\mathbf{x} \in \text{conv}[V_0(S) - \pi(\mathbf{e}_n, V_1(S))] = H$. Thus $D = H$.

Now the set W of vectors

$$\mathbf{x}'_0 - \mathbf{x}'_{h+1}, \mathbf{x}'_1 - \mathbf{x}'_{h+1}, \dots, \mathbf{x}'_h - \mathbf{x}'_{h+1}, \mathbf{x}'_h - \mathbf{x}'_{h+2}, \dots, \mathbf{x}'_h - \mathbf{x}'_n$$

is an $(n-1)$ -dimensional set. For the dimension of W is that of the space spanned by the set

$$\mathbf{x}'_1 - \mathbf{x}'_0, \mathbf{x}'_2 - \mathbf{x}'_0, \dots, \mathbf{x}'_h - \mathbf{x}'_0, \\ (\mathbf{x}'_{h+1} - \mathbf{x}'_{h+2} + \mathbf{x}'_h - \mathbf{x}'_0), \dots, (\mathbf{x}'_{h+1} - \mathbf{x}'_n + \mathbf{x}'_h - \mathbf{x}'_0),$$

which is that of the space spanned by the set

$$\mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_2 - \mathbf{x}_0, \dots, \mathbf{x}_h - \mathbf{x}_0, \mathbf{x}_{h+1} - \mathbf{x}_{h+2}, \dots, \mathbf{x}_{h+1} - \mathbf{x}_n,$$

which is at most one less than that of the space spanned by the set

$$\mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_2 - \mathbf{x}_0, \dots, \mathbf{x}_h - \mathbf{x}_0, \mathbf{x}_{h+1} - \mathbf{x}_0, \mathbf{x}_{h+2} - \mathbf{x}_0, \dots, \mathbf{x}_h - \mathbf{x}_0,$$

which is n .

Now $W \subset D$, and this finishes the proof of (1). Next we introduce the function

$$g(\mathbf{u}) = \mu_{n-1}[\pi(\mathbf{e}_n + \mathbf{u}, S)] = \mu_n(S)f(\mathbf{e}_n + \mathbf{u}, S),$$

where $\mathbf{u}$ is a variable vector in P_0 . By [3], this is a convex function of $\mathbf{u}$. By (1), Lemmas 2 and 3, and the convexity of g , it follows that $g(\mathbf{u}) = n\mu_n(S)$ on D . By (1), D has nonempty interior in P_0 and so, again using the convexity of g , it follows that

$$g(\mathbf{u}) \geq n\mu_n(S), \quad \mathbf{u} \in P_0.$$

Thus $f(\mathbf{a}, S) \geq n$, which proves the theorem.

4. Remarks. 1. It is possible to show that, in fact,

$$f(\mathbf{a}, S) > n \quad \text{whenever} \quad \pi(\mathbf{e}_n, \mathbf{a}) \in P_0 \sim D.$$

2. Rogers and Shephard [2] define a convex body K to be a pseudodouble pyramid if it is possible to find points $\mathbf{a}, \mathbf{b}$ on the boundary of K such that each 2-dimensional section of K by a plane through $\mathbf{a}$ and $\mathbf{b}$ is either a triangle with $\mathbf{a}$ and $\mathbf{b}$ as vertices, or a convex quadrilateral with $\mathbf{a}$ and $\mathbf{b}$ as opposite vertices. If we call $\mathbf{a} - \mathbf{b}$ the direction of the axis of such a body, it follows from [2] and the Theorem that for every direction ξ , a simplex is a pseudodouble pyramid with axis in that direction.

3. I believe that the simplex is the only class of extremals for the maximization problem discussed, but have been unable to prove this.

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