TITUS' HOMOTOPIES OF NORMAL CURVES

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ABSTRACT. A regular homotopy of a plane immersion of the circle, each of whose stages is a normal curve with the exception of a finite number of stages of the homotopy presenting a nonnegative convex double point self tangency or a transverse triple point, preserves the number of topologically inequivalent extensions of the immersion to an orientation preserving immersion of the disk. Extensions to properly interior mappings of the disk are similarly investigated.

1. Introduction. In this note we present a proof of a conjecture of C. J. Titus concerning regular homotopies of plane immersed circles that preserve certain bounding properties of the curves. Whitney [10, p. 276] defines a plane closed curve to have a *regular representation* if it can be parametrized by a differentiable immersion f of the oriented circle S into the oriented plane E . The immersion f is *normal* if its oriented image $[f]$ lies in general position. In that case, $[f]$ has but finitely many transverse double point self-intersections (*nodes*). By a regular homotopy h between two immersions f and g of S we shall mean a smooth map $h: I \times S \rightarrow E$, $I = [0, 1]$, each stage of which, h_t , is an immersion of S and $h_0 = f$, $h_1 = g$.

Recall that the oriented self-intersection pattern of a normal immersion, the *Titus intersection sequence* of f , is a complete topological invariant [9, p. 49]. Hence, a regular homotopy, each of whose stages h_t is normal, terminates at equivalent curves, i.e., there are orientation preserving self-homeomorphisms k_S of S and k_E of E such that $h_1 \circ k_S = k_E \circ h_0$. Elsewhere [2] we show that the stages of a regular homotopy "generically" present two further, nonnormal, self-intersections. That is, in every C^1 -neighborhood of a C^1 -smooth regular homotopy there is such a Titus homotopy (defined presently), and every C^2 -map of the annulus sufficiently C^2 -near a C^2 -smooth Titus homotopy is again such a regular homotopy. (Since the annulus is compact, we may use the topologies of uniform convergence in posi-

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tion and the first, resp. first and second partial derivatives.) A *Titus homotopy* h is a smooth regular homotopy such that for finitely many $t_j \in I \setminus \{0, 1\}$, the stage $[h_{t(j)}]$ has, in addition to nodes, a single convex double point tangency or a single transverse triple point. Up to a local isotopy, the latter two events can be pictured in the Cartesian x, y -plane by the intersection at the origin $x=0, y=0$, at time $s=0$, of a moving arc parametrized by s with one, resp. two stationary arcs. For example:

Type I. Node: $x=0, y=0$;

Type II. Convex tangency: $x=0, x=y^2 \pm s$;

Type III. Triple point: $x=y, x=-y, x=\pm s$.

The remaining stages have only nodes. The orientation of the segments further subdivides the types. The sign of the s -parameter indicates which way the homotopy moves. Observe that for Type II, a bilateral cell is either created or destroyed, see Figure 2. If the oriented edges of the bilateral form a counterclockwise cycle, we speak of Type II⁺; if clockwise, it is Type II⁻, and if the edges are opposed, it is Type II⁰. For Type III, Figure 1, one trilateral goes over into another. If the first is cyclicly oriented, so is the second, but with opposite orientation, and we speak of Type III[±]. We collect the noncyclic cases into the mixed Type III⁰.

Clearly such homotopies are combinatorially reflected by the corresponding transformation on the intersection sequence. It serves no present purpose to describe these in detail. Of more immediate interest is to investigate what bounding properties of plane immersions are preserved by each type.

Titus conjectured

THEOREM 1. *A regular homotopy of the above kind that does not have any negatively oriented convex double point tangencies (Type II⁻) preserves the number of topologically inequivalent extensions of the immersion to an orientation preserving immersion of the disk.*

REMARK 1. An immersion of the disk in the plane is a special case of a properly interior map of the disk. Recall [9, p. 46] that a continuous map F of the closed disk $\bar{D}$ to the plane which is light, open and orientation preserving on the open disk D and a local homeomorphism on the boundary ∂D is topologically equivalent to an analytic map. Hence a branch point, where F fails to be locally one-to-one, is isolated, and F is a $k:1$ covering map on a deleted neighborhood of the point. We prove the lemmas for Theorem 1 in the more general setting of properly interior extensions because these are inherently more interesting. However, the difficulty encountered as the

homotopy sweeps across branch points discourages a present formulation of the analog to Theorem 1. To simplify the proof of Lemma 1 we assume throughout the paper that a *properly interior* map F is continuous on $\overline{D}$, light-open orientation preserving on D , and a local homeomorphism on all of $F^{-1}(F(\partial D))$.

2. Definitions. Let $h: f \simeq g$ be a homotopy between two normal immersions of S in E . Let A' be an open subarc of S on which f is univalent and whose extremities, X' and Y' , do not map to nodes of $[f]$. Let $A'' = S \setminus \overline{A}'$. (We reserve capitals A, B, C for open arcs, D for the open disk, E for the plane, F, G, H for maps from disks to the plane, J for a Jordan loop, K through Q and X, Y, Z for points, and S for the circle.) The Titus homotopy is said to be *atomic* if

- (i) $f|_{\overline{A}''} = h_t|_{\overline{A}''}$ for all $t \in I$,
- (ii) $h|_{I \times A'}$ is an embedding, and
- (iii) for exactly one value of t , $[h_t|_{A'}]$ meets $[h_t|_{A''}]$ at either a single transverse triple point or a single convex double point tangency.

The homotopy is said to *grow* from f to g if the embedding is orientation preserving, otherwise it *shrinks* from f to g .

Figures 1, 2 and 3 represent a detail of the situation near the exceptional self-intersection. In each, the solid lines are part of $[f]$, $[f|_{\overline{A}'}]$ is the oriented arc $XMNY$, and $[g|_{\overline{A}'}]$ the dashed arc $KKLY$. In Figure 1, $[f]$ has nodes M, N, P ; $[g]$ has nodes K, L, P ; the triple point is at P . In Figures 2 and 3, the $[f]$ -nodes are M, N and are deleted by the homotopy. Note that the homotopy in Figures 1 and 2 shrinks, and in Figure 3 grows f to g . Figure 1 will serve for all Types III, Figure 2 for Type II^0 and II^+ while Figure 3 is the forbidden Type II^- .

The strategy of proof is to lift the dashed arc via any properly interior extension F of f to the disk. If F has no branch points "belonging" to the cell MNP , a fortiori if F is an immersion, the lift is mapped univalently by F onto its image. Since the source is a disk, the lift divides D into two subdisks, one of which is embedded onto the cell, and so may be "subtracted" from F .

3. Lifting arcs. We restate M. Marx' arc-lifting theorem [5, Theorem 1, p. 49] suitable for our context.

THEOREM 2 (MARX). *Let $F: \overline{D} \rightarrow E$, $F|_{\partial D} = f$, be properly interior. Let $e: I \rightarrow E$ be an embedded arc with $e(0) = F(Z')$, $Z' \in \overline{D}$. If $Z' \in D$ or if $Z' \in \partial D$ and $[e]$ lies to the left of $[f]$ at $f(Z')$ (i.e. the angle α from $df(Z')/d\theta$ to $de(0)/ds$ satisfies $0 < \alpha < \pi$), then there exists an arc $B \subset D$*

beginning at Z' , ending at Z'' , such that $F|_{\overline{B}}$ is a homeomorphism of $\overline{B}$ into $[e]$, and either $Z'' \in \partial D$ or $Z'' \in D$ and $F(Z'') = e(1)$.

DEFINITION. We say such an arc B lifts $[e]$ from Z' .

For the following four lemmas we assume that we have the same situation as depicted by Figure 1 and Figure 2 only.

LEMMA 1. If F is a properly interior extension of f to the disk, there exists a subdisk $D_0 \subset D$ such that

$$F|_{\overline{D_0}}: (D_0, \partial D_0) \rightarrow (\text{Ins}(MQNPM), MQNPM),$$

is a light open map of the disk D_0 onto the cell enclosed by the Jordan loop $MQNPM$, which is a $(k+1)$ -fold covering immersion on the boundary, where k is the sum of the multiplicities of the branch points of $F|_{D_0}$.

PROOF. We borrow Marx' argument for lifting the monohedral cell formed by a positive terminal loop [6, Lemma 3, p. 83] and we present only the argument for the triple point (Figure 1). The point Q has f -preimage $Q' \in \partial D$. The nodes M, N, P have two f -preimages on ∂D , labeled so that $M'Q'N'$ maps to MQN , $N''P'$ to NP , and $P''M''$ to PM . Set $B_0 = M'Q'N'$. Now $N' \in D$ and the arc NP lies to the left of $[f]$ at N . Let B_1 lift NP from N' . Then B_1 ends in some $P_1 \in F^{-1}(P)$. Either $P_1 \in \partial D$ or $P_1 \in D$. In the former case $P_1 = P''$ and $[f]$ is oriented LPM because F , being properly interior, is an orientation preserving local homeomorphism on ∂D . Set $B_2 = P''M''$ and $M'' = M_2$. Since $M_2 \in \partial D$ and MQN lies to the left of $[f]$ at M , let B_3 lift MQN from M_2 ending with $N_2 \in F^{-1}(N)$. In the latter case, $P_1 \in D$, let B_2 lift PM from P_1 ending in $M_2 \in F^{-1}(M)$. Either $M_2 \in \partial D$ or $M_2 \in D$. In the former case $M_2 = M'$ and we stop with B_2 and $k=0$. In the latter case, $M_2 \in D$, we lift MQN to B_3 from M_2 ending at N_3 .

Now, whichever alternative led to B_3 , either $N_3 \in \partial D$ or $N_3 \in D$. In the former case $N_3 = N''$ and $[f]$ is oriented NPL . Set $B_4 = N''P'$ and $P' = P_4$. Since $P_4 \in \partial D$ and PM lies to the left of $[f]$ at P we lift PM to B_5 from P_4 ending at M_5 . In the latter case, $N_3 \in D$, let B_3 lift NP from N_3 , etc. We continue in this fashion, stopping only if some B_{3k+2} ends in M' .

Since $J = \overline{B_0} + \overline{B_1} + \cdots$ lies in $F^{-1}[f]$ and F is a local homeomorphism there, J never intersects itself. Hence J must terminate at M' with some arc B_{3k+2} . So J is a Jordan loop, circumscribing positively a subdisk D_0 of D . Further, $F|_{\partial D_0}$ is a (piecewise) regular map of J onto the Jordan loop $MQNPM$ with degree $k+1$. By a theorem of M. Morse [7, Theorem 20.1, p. 71] the tangent winding number

(=Gaussian degree) of the boundary immersion of a properly interior map of the disk exceeds by one the sum of the branch point multiplicities of the map. Thus the map $F|D_0$ has branch points of total multiplicity k whose images reside in the cell MNP .

LEMMA 2. *If, under the hypotheses of Lemma 1, $k=0$, then $F|D_0$ is an embedding. If $XKLY$ is drawn sufficiently near MPN then it can be lifted to a simple arc C in D with $\overline{C} \cap \partial D = \{X', Y'\}$. Let D_1 be the sub-disk of D with $\partial D_1 = \overline{C} + \overline{A}$, where $A = X'Q'Y'$ is homeomorphic to XQY under f . If D_2 is the complementary disk to D_1 then $G = F|D_2$ is a properly interior extension of $g = G|\partial D_2$.*

PROOF. Note that if the arc J in the proof of Lemma 1 touches ∂D except on $\overline{B}_0$ then $k \geq 1$. So if $k=0$, $J \setminus B_0 = \overline{B}_1 + \overline{B}_2$ meets ∂D only at its endpoints. Thus the complement of $\overline{D}_0$ is also a disk, say D' . (This need not be the case if we had lifted a polyhedral cell of more than three sides.) Now $F|D'$ is properly interior and $F|\overline{D}_0$ a homeomorphism. Let $XKLY$ be so near MPN that $F|D'$ lifts it to a simple arc C that meets ∂D at the endpoints X' and Y' . Let $A = X'Q'Y'$ in ∂D and D_1 the disk cut by C with $\partial D_1 = \overline{A} + \overline{C}$. If D_2 is the complementary disk then $G = F|D_2$ is still properly interior, and an extension of $g = G|\partial D_2$.

LEMMA 3. *If g extends to a properly interior map G of D , then so does f with no further creation of branch points.*

PROOF. The arc $XMNY$ is in both cases a simple positive detour of $[g]$. See [1, p. 278]. Since G is properly interior, we may consider G regular in an annulus adjacent to D . Thus, treating the restriction of G to this annulus as a positive monotone homotopy, we may add the detour to it to form a monotone homotopy ending with $[f]$. (All this is essentially a Schoenflies type argument.) The resulting map F is properly interior, with $F|\partial D = f$. But f has the same tangent winding number as g . Hence, by Morse's theorem quoted above no branch points can have been added.

LEMMA 4. *Let F and F^* be two extensions of f to properly interior maps of $\overline{D}$. Suppose the integers k and k^* from Lemma 1 are both zero, and suppose that $G = F|\overline{D}_2$ and $G^* = F^*|\overline{D}_2^*$ are topologically equivalent. Then so are F and F^* .*

PROOF. Let $H: \overline{D}_2^* \rightarrow \overline{D}_2$ realize the equivalence, i.e., $F \circ H = F^*|\overline{D}_2^*$. Extend H to $\overline{D}_1^*$ by setting $H|\overline{D}_1^* = F^{-1} \circ F^*|\overline{D}_1^*$. Then $F \circ H = F^*$ on $\overline{D}$.

FIGURE 1

FIGURE 2

FIGURE 3

4. Proof of Theorem 1. Under the hypotheses, the Titus homotopy h may be decomposed into a succession of homotopies, $h'_1 + h''_1 + h'_2 + h''_2 + \cdots + h'_m$, where each h'_j passes through normal stages only, and each h''_j is, up to an isotopy of E , an atomic homotopy of Types III or Types II^+ and II^0 . Since neither an isotopy of E , nor a normal homotopy can alter the number of topologically inequivalent extensions to an immersion of the disk, we need to prove the theorem only for atomic homotopies.

Suppose then, that h is a shrinking atomic homotopy from f to g . Then to each extension of f , Lemma 1 supplies $k=0$, and Lemma 2 supplies the extension of g . By Lemma 3 an extension of g gives rise to an extension of f . Lemma 4 says that this relation is one-to-one up to topological equivalence. Should $h:f \simeq g$ be a growing atomic homotopy, the map $\bar{h}(t, \theta) = h(1-t, \theta)$ is a shrinking homotopy from g to f . Reversing the roles of f and g we argue as above.

REMARK 2. The tangent winding number of a plane immersion of the circle is, by the Whitney-Graustein Theorem [10], a regular homotopy invariant. By Morse's theorem the tangent winding number of f must be at least 1 in order for f to have any properly interior extension. Moreover, f must have nonnegative winding number about every point of E [9, p. 48]. To see how the forbidden Type II^- homotopy affects things, complete Figure 3 by joining the arc ends at the top and those at the bottom. Thus $[f]$ has winding numbers $+1$ and -1 and so no extension, while $[g]$, a positive Jordan loop, has exactly one extension.

The effect of $k > 0$ can be seen by completing Figure 1 to the trefoil $XMQNYLPMNPKX$. Here $[f]$ has tangent winding number 2, and

so has a unique extension to a properly interior map with a unit multiplicity branch point in the trihedral cell. The homotopy is of Type III⁺, and so $[g]$ has a trihedral cell of negative circulation and $[g]$ cannot extend.

We complete Figure 2 by continuing MQN past Y , through nearly 360° counterclockwise to join before N , and close the figure by taking NPM around to join before X . Here $[f]$ has tangent winding number three and bounds a properly interior map with either two branch points of multiplicity 1 each, or a single, doubly-multiple branch point in the dihedral cell. This Type II⁺ homotopy shrinks f to g . Now g bounds a properly interior map with simple branch points in each of the two terminal loops. Here the number of inequivalent extensions is not preserved.

REMARK 3. These examples show that the analog to Theorem 1 in the case of tangent winding number greater than one is considerably more difficult. We call the reader's attention to A. Gramain's recent results [4]. He shows, in particular, that every immersion of S in E with tangent winding number $(1-2g)$, g a positive integer, is regularly homotopic to one that extends to an immersion of a disk of genus g (i.e. a compact, oriented two-manifold with a disk removed). For example, the normal immersion $[f]$ with Titus intersection sequence

$$\begin{array}{cccccccccc} + & + & - & + & - & + & - & + & - & - \\ 0 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 1 & 0 \end{array}$$

bounds an immersed disk of genus one. (See [1, p. 288] for this notation.) The connected sum in E of g copies of $[f]$ bounds an immersed disk of genus g . It would be interesting to know which Titus homotopies preserve extendability of plane immersions of disks of higher genus.

The problem of finding the exact number of inequivalent extensions of an immersion S in E to an immersion of the disk has been completely solved by S. Blank [8]. (See also [3, p. 378] for a brief outline and application of Blank's theorem.) These methods are in the process of being refined by M. Marx to apply also to the problem of inequivalent properly interior extensions. There remains the question of whether an immersion with k different extensions is homotopic to Blank's canonical curve for k [8] under a Titus homotopy that has no tangencies of Type II⁻. While there is evidence to the contrary, this question is unresolved.

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