CYCLIC ATOMS IN ORTHOMODULAR LATTICES¹

DONALD E. CATLIN

ABSTRACT. Let $P(H)$ denote the projection lattice of a separable Hilbert space H . For each $\mathbf{x} \in H$, let $P_{\mathbf{x}}$ denote the projection onto the one dimensional subspace generated by $\mathbf{x}$. If B is a Boolean sublattice of $P(H)$, then it is a theorem that whenever B is maximal in $P(H)$ there exists a vector $\mathbf{x}_0 \in H$, called a *cyclic vector for B* , such that the join in $P(H)$ of all the $P_{Q(\mathbf{x}_0)}$ as Q ranges through B is the identity operator I . In this paper we show that this theorem is an immediate corollary of a more general theorem in orthomodular lattice theory. In addition, a final theorem in the paper makes clear the necessity for the separability assumption on H .

1. Introduction. Let $P(H)$ denote the projection lattice of a separable Hilbert space H . For each $\mathbf{x} \in H$, let $P_{\mathbf{x}}$ denote the projection onto the one dimensional subspace generated by $\mathbf{x}$. If B is a Boolean sublattice of $P(H)$, then it is a theorem that whenever B is maximal in $P(H)$ there exists a vector $\mathbf{x}_0 \in H$, called a *cyclic vector for B* , such that the join in $P(H)$ of all the $P_{Q(\mathbf{x}_0)}$ as Q ranges through B is the identity operator I . Although the converse of this theorem holds, it is the theorem as stated that is of interest in applications ([6] and [7]). In an address to the 1968 NSF Conference on Quantum Mechanics and Orthomodular Lattices held at the University of Massachusetts, R. J. Greechie posed the problem of finding a lattice theoretic analog of this theorem. In this paper we present a solution to Greechie's problem.

2. Preliminaries. By an *orthomodular lattice* we mean an orthocomplemented lattice $(L, \leq, ')$, $'$ denoting the orthocomplement, such that whenever $x, y \in L$ with $x \leq y$, the relation $y = x \vee (y \wedge x')$ holds. Henceforth L will always represent an orthomodular lattice and any seemingly homeless elements will belong to L . Two elements $x, y \in L$ are said to be orthogonal, written $x \perp y$, provided that $x \leq y'$. Two elements $x, y \in L$ are said to *commute*, written $x C y$, in case $x = (x \wedge y) \vee (x \wedge y')$. For each $a \in L$ we define a mapping $\phi_a: L \rightarrow L$ by the relation

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$$\phi_a(x) = (x \vee a') \wedge a.$$

This mapping is called the *Sasaki projection on a*. The basic facts on commutativity and Sasaki projections are given in [2, Lemmas 1 and 2, pp. 66–67]. Since some of these facts are used in this paper, we quote these lemmas here:

2.1 LEMMA. Let $e, f \in L$ and $\{e_\alpha \mid \alpha \in A\} \subset L$. Then

- (i) $(\phi_e)^2 = \phi_e$,
- (ii) $\phi_e(\phi_e(f)') = f' \wedge e$,
- (iii) if $\bigvee_\alpha e_\alpha$ exists, then $\bigvee_\alpha \phi_e(e_\alpha)$ exists and $\phi_e(\bigvee_\alpha e_\alpha) = \bigvee_\alpha \phi_e(e_\alpha)$,
- (iv) $\phi_e(f) = f \Leftrightarrow f \leq e$,
- (v) $\phi_e(f) = 0 \Leftrightarrow f \leq e'$, i.e., $f \perp e$.

2.2 LEMMA. Let $e, f \in L$. Then

- (i) $e C f \Rightarrow f C e$,
- (ii) $e C f \Leftrightarrow \phi_f \phi_e = \phi_f \phi_e \Leftrightarrow \phi_e \phi_f = e \wedge f$,
- (iii) $e C f \Leftrightarrow (e \vee f') \wedge f = e \wedge f \Leftrightarrow (f \vee e') \wedge e = f \wedge e$,
- (iv) $e C f \Rightarrow e' C f$.

In addition to these lemmas, it is helpful to observe that if x and y are either comparable or orthogonal then $x C y$. Also, by 2.2 (iii), $e C f \Leftrightarrow \phi_f(e) = e \wedge f$, and this latter statement is equivalent to $\phi_f(e) \leq e \wedge f$ since it is always true that $\phi_f(e) \geq e \wedge f$.

A subset B of an orthomodular lattice $(L, \leq, ')$ is called a *Boolean sublattice* provided that B is a sublattice of L ; B is suborthocomplemented, i.e., $x \in B \Rightarrow x' \in B$; and $x, y \in B \Rightarrow x C y$. Applying [2, Lemma 4, p. 67], it is clear that a Boolean sublattice is certainly a Boolean lattice in the usual sense. If B is a maximal Boolean sublattice, then B is called a *block*, the terminology due to Greechie [3]. If for $M \subset L$ we define $C(M) = \{x \mid x C y \ \forall y \in M\}$, it follows from Lemmas 2.1 and 2.2 that $C(M)$ is a sublattice of L . Moreover, B is a block if and only if $B = C(B)$. In case $(L, \leq, ')$ is complete, one can prove that $C(M)$ is a subcomplete sublattice of L (use [2, Lemma 3, p. 67]).

A nonzero element a in an orthomodular lattice $(L, \leq, ')$ is called an *atom* in case $0 \leq x \leq a$, $x \neq 0 \Rightarrow x = a$. $(L, \leq, ')$ is called *atomic* providing every nonzero element has an atom under it. Using orthomodularity one can show that if $(L, \leq, ')$ is atomic it is also *atomistic*, i.e., every element is the join of the atoms under it. Finally $(L, \leq, ')$ is said to have the *atomic projection property* if whenever a is an atom, $\phi_b(a)$ is an atom or zero for every $b \in L$.

3. The hyperoctant property. Let $(L, \leq, ')$ be an atomic ortho-

modular lattice. Then L is said to have the *atomic bisection property* (ABP) provided that for every pair of disjoint atoms $a, b \in L$ there exists an atom $c \in L$ such that $c \leq a \vee b$, $c \neq a$ and $c \neq b$ ([1] and [4]). It is easy to show that ABP is the same as requiring that for every pair of orthogonal atoms $a, b \in L$ there exists an atom c such that $c \leq a \vee b$, $c \not\leq a$ and $c \not\leq b$ ($\not\leq$ means "does not commute with"). With this motivation we define $(L, \leq, ')$ to have the *hyperoctant property* (HP) if it is the case that for every orthogonal family of atoms $\{a_\alpha\}$ in L , with cardinality of $\{a_\alpha\} \geq 2$, there exists an atom a such that $a \leq \bigvee_\alpha a_\alpha$ and $a \not\leq a_\alpha$ for each α . In case it is necessary for the orthogonal set $\{a_\alpha\}$ to be finite (or countable) to assure existence of the aforementioned a , we say that L has the *finite hyperoctant property* (FHP) (or σ -hyperoctant property (σ -HP)). Clearly (using Fourier expansions) one can show that the projection lattice of any Hilbert space has the σ -HP and also if the space fails to be separable, then it does not have the HP. Obviously $\text{HP} \Rightarrow \sigma\text{-HP} \Rightarrow \text{FHP} \Rightarrow \text{ABP}$ and by the above remark the first of these implications cannot, in general, be reversed. Whether or not $\text{FHP} \Rightarrow \sigma\text{-HP}$ is unknown at this time. We will devote the remainder of this section to showing that $\text{ABP} \Rightarrow \text{FHP}$.

3.1 LEMMA. Let $a, b, c \in L$, b an atom, $c \perp b$, a not orthogonal to b , and $a \leq b \vee c$. Then $a \vee c = b \vee c$.

PROOF. $a \vee b \leq b \vee c$ so $a \vee c = (a \vee c) \wedge (b \vee c) = ((a \vee c) \wedge b) \vee c$ (orthomodularity). But $(a \vee c) \wedge b = 0$ is not possible since $c \neq a \vee c$ and hence $(a \vee c) \wedge b = b$ because b is an atom. Thus $a \vee c = b \vee c$.

3.2 THEOREM. $\text{ABP} \Rightarrow \text{FHP}$.

PROOF. Let $\{a_i \mid i=1, 2, \dots, n\}$, $n \geq 2$, be an orthogonal family of atoms. We wish to show that there is an atom a such that $a \leq \bigvee_{i=1}^n a_i$, $a \not\leq a_i$ for each $i=1, 2, \dots, n$. The proof is by induction on n . Certainly if $n=2$, ABP implies the existence of such an a . Suppose the statement is true for some $n \geq 2$ and let $\{a_i \mid i=1, 2, \dots, n+1\}$ be an orthogonal family of atoms. By the induction hypothesis there exists an atom a such that $a \leq \bigvee_{i=1}^n a_i$ and $a \not\leq a_i$ for $i=1, 2, \dots, n$. Since $a' \geq \bigwedge_{i=1}^n a_i' \geq a_{n+1}$, a_{n+1} is orthogonal to a . By ABP, choose b so that $b \leq a \vee a_{n+1}$, $b \not\leq a$ and $b \not\leq a_{n+1}$. Suppose $b \leq a_j$ for some $j \leq n$. Then since b and a_j are atoms, either $b = a_j$ or $b \leq a_j'$. If $b = a_j$, then $a_j \not\leq a_{n+1}$, a contradiction. Thus $b \leq a_j'$. Now $b \leq a \vee a_{n+1}$, $a \perp a_{n+1}$, b is not orthogonal to a , and a is an atom, so by Lemma 3.1 we have $b \vee a_{n+1} = a \vee a_{n+1}$, whence $a \leq b \vee a_{n+1}$. Now $b \leq a_j'$ and $a_{n+1} \leq a_j'$ so that $a \leq a_j'$, a contradiction since $a \not\leq a_j$. Hence $b \not\leq a_i$ for $i=1, 2, \dots, n, n+1$ and $b \leq \bigvee_{i=1}^{n+1} a_i$.

4. The residuated envelope of a Boolean sublattice. Let $B \subset L$ be a Boolean sublattice of the complete orthomodular lattice L . By the *residuated envelope* of B , denoted Ψ_B , we mean the map $\Psi_B: L \rightarrow L$ defined by $\Psi_B(x) = \bigvee_{b \in B} \phi_b(x)$. Ψ_B is an example of a *residuated map*. Though we will not make use of this fact, it is easily established and thus we include its proof as part of (vi) in the following theorem. For the definition of a residuated map see [5].

4.1 THEOREM. (i) $\Psi_B(0) = 0$.

(ii) $x \leq \Psi_B(x)$.

(iii) $x \leq y \Rightarrow \Psi_B(x) \leq \Psi_B(y)$.

(iv) $\Psi_B^2 = \Psi_B$.

(v) Ψ_B is a complete join homomorphism.

(vi) Ψ_B is a residuated closure operator (see [5]).

PROOF. (i) is clear.

(ii) Since $1 \in B$, $x = \phi_1(x) \leq \bigvee_{b \in B} \phi_b(x) = \Psi_B(x)$.

(iii) If $x \leq y$, then for every $b \in B$, $\phi_b(x) \leq \phi_b(y)$ and the result follows.

(iv) The proof makes use of 2.1 (iii) and 2.2 (ii).

$$\begin{aligned} \Psi_B^2(x) &= \bigvee_{b \in B} \phi_b(\Psi_B(x)) = \bigvee_{b \in B} \phi_b \left(\bigvee_{c \in B} \phi_c(x) \right) = \bigvee_{b \in B} \bigvee_{c \in B} \phi_b \circ \phi_c(x) \\ &= \bigvee_{b \in B} \bigvee_{c \in B} \phi_{b \wedge c}(x) = \bigvee_{b \in B} \phi_b(x) = \Psi_B(x). \end{aligned}$$

(v) This follows at once from 2.1 (iii).

(vi) To establish that Ψ_B is residuated it suffices to check that for each $y \in L$, the (nonempty) set $\{x \in L \mid \Psi_B(x) \leq y\}$ has a greatest element. But this follows easily from (v). Ψ_B is a closure operator by virtue of (ii) and (iv).

4.2 THEOREM (i) If $c \in C(B)$, then $\Psi_B(\phi_c(x)) = \phi_c(\Psi_B(x))$.

(ii) If $b \in B$, then $b \leq \Psi_B(x)$ for each $x \in L$.

(iii) $\Psi_B(x) = x$ if and only if $x \in C(B)$.

PROOF.

$$\begin{aligned} \phi_c(\Psi_B(x)) &= \phi_c \left(\bigvee_{b \in B} \phi_b(x) \right) = \bigvee_{b \in B} \phi_c \circ \phi_b(x) \\ \text{(i)} \quad &= \bigvee_{b \in B} \phi_{b \circ c}(x) = \Psi_B(\phi_c(x)). \end{aligned}$$

(ii) If $b \in B$, then

$$\begin{aligned}
 \phi_b(\Psi_B(x)) &= \Psi_B(\phi_b(x)) = \bigvee_{c \in B} \phi_c \phi_b(x) = \bigvee_{c \in B} \phi_{c \wedge b}(x) \\
 &= \bigvee_{c \in B; c \leq b} \phi_c(x) \leq \bigvee_{c \in B} \phi_c(x) = \Psi_B(x).
 \end{aligned}$$

This is enough to imply commutativity.

(iii) If $x \in C(B)$, then $x \leq \Psi_B(x) = \Psi_B(\phi_x(x)) = \phi_x(\Psi_B(x)) \leq x$. Hence $\Psi_B(x) = x$. Conversely, if $\Psi_B(x) = x$, then by part (ii), $x \leq b$ for every $b \in B$ and so $x \in C(B)$.

4.3 COROLLARY. $\text{Range } \Psi_B = C(B)$.

4.4 THEOREM. *The following are equivalent:*

- (i) $\Psi_B(x) \perp y$.
- (ii) $x \perp \Psi_B(y)$.
- (iii) $\Psi_B(x) \perp \Psi_B(y)$.

Moreover, these imply $x \perp y$

PROOF. (i) $\Leftrightarrow$ (ii). $\Psi_B(x) \perp y \Leftrightarrow \phi_b(x) \leq y'$ for every $b \in B$. Now $\phi_b(x) \leq y'$ implies that $y \leq (x' \wedge b) \vee b'$. Thus $y \vee b' \leq (x' \wedge b) \vee b'$ whence $\phi_b(y) \leq x' \wedge b \leq x'$. By symmetry, $\phi_b(x) \leq y' \Leftrightarrow \phi_b(y) \leq x'$.

(i) $\Leftrightarrow$ (iii). $\Psi_B(x) \perp y \Leftrightarrow \Psi_B^2(x) \perp y \Leftrightarrow \Psi_B(x) \perp \Psi_B(y)$.

The last statement follows from 4.1 (ii).

5. The cyclic atom property. Let $(L, \leq, ')$ be a complete atomic orthomodular lattice. We say that L has the *cyclic atom property* if whenever B is a Boolean sublattice such that $C(B)$ is a block, then there exists an atom x_0 such that $\Psi_B(x_0) = 1$. Such an atom is called a *cyclic atom* for B .

5.1 THEOREM. *If L has the hyperoctant property then L has the cyclic atom property.*

PROOF. Let B be a Boolean sublattice such that $C(B)$ is a block. Let $\{a_\alpha\}$ be a family of atoms in L maximal with respect to the property that $\{\Psi_B(a_\alpha)\}$ is an orthogonal family. Then $\{a_\alpha\}$ is an orthogonal family. If $\{a_\alpha\}$ contains only one atom, say b , we claim $\Psi_B(b) = 1$. For if not, then $\Psi_B(b)' \neq 0$ and we can find an atom $c \leq \Psi_B(b)'$, whence $c \perp \Psi_B(b)$. Thus $\Psi_B(c) \perp \Psi_B(b)$ contradicting maximality of $\{b\}$. Thus we can suppose $\{a_\alpha\}$ contains at least two atoms. Using HP, choose x_0 so that $x_0 \leq \bigvee_\alpha a_\alpha$ and $x_0 \not\leq a_\alpha$ for every α . Now for each β , $x_0 \leq a_\beta \vee (\bigvee_{\alpha \neq \beta} a_\alpha)$. Now $a_\beta \perp (\bigvee_{\alpha \neq \beta} a_\alpha)$, a_β is an atom, and x_0 is not orthogonal to a_β , so by 3.1 we have $x_0 \vee \bigvee_{\alpha \neq \beta} a_\alpha = a_\beta \vee \bigvee_{\alpha \neq \beta} a_\alpha$. Applying Ψ_B we obtain

$$\Psi_B(x_0) \vee \bigvee_{\alpha \neq \beta} \Psi_B(a_\alpha) = \Psi_B(a_\beta) \vee \bigvee_{\alpha \neq \beta} \Psi_B(a_\alpha).$$

Now since $\text{range } \Psi_B = C(B)$, which by hypothesis is a block, we have the distributive law at our disposal. Meeting both sides of the above expression with $\Psi_B(a_\beta)$ we obtain

$$\Psi_B(x_0) \wedge \Psi_B(a_\beta) = \Psi_B(a_\beta)$$

i.e., $\Psi_B(a_\beta) \leq \Psi_B(x_0)$. Since this holds for all β , we have $\bigvee_\beta \Psi_B(a_\beta) \leq \Psi_B(x_0)$. We claim that $\bigvee_\beta \Psi_B(a_\beta) = 1$. For if not, $0 \neq \bigwedge_\beta \Psi_B(a_\beta)'$ and so there exists an atom a such that $a \leq \Psi_B(a_\beta)'$ for every β . Thus $a \perp \Psi_B(a_\beta)$ for every β and hence $\Psi_B(a) \perp \Psi_B(a_\beta)$ for every β . By maximality $a = a_\beta$ for some β , whence $\Psi_B(a) \perp \Psi_B(a)$. It follows that $\Psi_B(a) = 0$, and hence $a = 0$, a contradiction. This completes the proof.

5.2 COROLLARY. *If L has the hyperoctant property, then every block has a cyclic atom.*

5.3 COROLLARY. *If H is a separable Hilbert space, then $P(H)$ has the cyclic atom property.*

5.4 THEOREM. *If L has the atomic projection property, then HP is equivalent to the cyclic atom property.*

PROOF. We need only show that the cyclic atom property implies HP. Let $\{a_\alpha\}$ be an orthogonal family of atoms with cardinality at least 2. Let B be any block such that $\{a_\alpha\} \subset B$. Choose x_0 to be a cyclic atom for B . Define $a = \bigvee_\alpha a_\alpha$ and let $b = \phi_a(x_0)$. Then b is an atom or zero. If $b = 0$, then $a \perp x_0$ and since $\Psi_B(a) = a$ we have $\Psi_B(a) \perp x_0$. Then by 4.4 we have $a \perp \Psi_B(x_0)$, whence $a = 0$, a contradiction. Thus b is an atom. If $b \leq a_\alpha$ for some α , either $b = a_\alpha$ or $b \perp a_\alpha$. If $b = a_\alpha$, then $\Psi_B(b) = \Psi_B(a_\alpha) = a_\alpha$. But $\Psi_B(b) = \Psi_B(\phi_a(x_0)) = \phi_a(\Psi_B(x_0)) = \phi_a(1) = a$, whence $a = a_\alpha$. This implies that the cardinality of $\{a_\alpha\}$ is one, a contradiction. If $b \perp a_\alpha$, then $b \perp \Psi_B(a_\alpha)$, so that $\Psi_B(b) \perp a_\alpha$. By preceding calculation, this means that $a \perp a_\alpha$, another contradiction. Thus $b \leq \bigvee_\alpha a_\alpha$ and $b \not\leq a_\alpha$ for each α , as required.

5.5 COROLLARY. *A Hilbert space H has the cyclic atom property if and only if it is separable.*

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UNIVERSITY OF MASSACHUSETTS, AMHERST, MASSACHUSETTS 01002