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QUICKLY OSCILLATING SOLUTIONS OF AUTONOMOUS ORDINARY DIFFERENTIAL EQUATIONS

STEPHEN R. BERNFELD AND A. LASOTA[1]

ABSTRACT. We are concerned here with the asymptotic behavior of quickly oscillating solutions of systems of differential equations. It is shown that the limit of the norm of any quickly oscillating solution exists and is either equal to infinity or zero. We then determine asymptotic bounds on the solutions by imposing certain growth conditions on the right-hand side of the equation. Our results, when applied to second order equations, yield asymptotic behavior of both the solutions and its derivatives.

1. **Introduction.** We are concerned here with the behavior of quickly oscillating solutions of a system of ordinary differential equations

$$x' = F(x), \tag{O}$$

where $F:R^n \to R^n$. We derive corollaries for the nth order scalar differential equation

$$x^{(n)} = f(x, x', x'', \cdots, x^{(n-1)}), \tag{N}$$

where $f:R^n \to R$. Special results are then obtained for the second order scalar differential equation

$$x'' = g(x, x'), \tag{S}$$

where $g:R^2 \to R$.

One example of quickly oscillating solutions arises from the following physical problem suggested to us by J. Yorke. Assume that a ball is dropped through air onto a surface such that the collision is elastic and, moreover, assume that the coefficient of friction in air is constant. We observe that the motion of the ball is quickly oscillating; that is, the time elapsed between collisions approaches zero. If we denote the displacement of the ball above the surface by $y(t)$ then $y(t)$ can be written as $y(t) = |x(t)|$, where $x(t)$ satisfies

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(P) $$x'' = -g \operatorname{sgn} x - vx',$$

in which v is the coefficient of friction in air and g is the gravitational constant. One rather surprising result we obtain is that if the amplitude of the motion of the ball is bounded then the force on it cannot be a continuously differentiable function.

The reader is referred to [1], [3] and [4] for other information on this subject.

2. **Results.** A function $x(t)$, where $x: R \to R$, is said to be quickly oscillating if it is defined in a neighborhood of $+\infty$ and if there exists a sequence of points $\{t_i\}$ such that

(1) $$x(t_i) = 0, \qquad i = 1, 2, \cdots,$$

and

(2) $$t_{i+1} > t_i, \qquad \lim_{i\to\infty} t_i = +\infty, \qquad \text{and} \qquad \lim_{i\to\infty} (t_{i+1} - t_i) = 0.$$

An n-vector function $x = (x_1, \cdots, x_n)$ is said to be quickly oscillating if every component x_j is a quickly oscillating function.

A function $\phi: R^n \to R^m$ is said to be locally bounded if ϕ is bounded in every bounded subset of its domain. Throughout this paper we shall assume that the functions $F: R^n \to R^n$, $f: R^n \to R$ and $g: R^2 \to R$ are locally bounded. We shall assume solutions of equation (O) ((S), (N)) are differentiable everywhere. If $x = (x_1, x_2, \cdots, x_n) \in R^n$ we define the norm of x as $\|x\| = \sum_{i=1}^n |x_i|$.

In this paper we show that the norm of any quickly oscillating solution of (O) either approaches zero or infinity as t tends to infinity. By placing certain growth conditions on F we conclude that every quickly oscillating solution must approach zero in the limit. Moreover, if F satisfies a Lipschitz condition at the origin we then have that the only bounded quickly oscillating solution is the trivial one, $x(t) \equiv 0$. Our results, when applied to equations (N) and (S), include generalizations of the results in [1], [3] and [4], as well as new information concerning the behavior of quickly oscillating solutions and their derivatives.

We now state our results with the proofs following later.

THEOREM 1. *If x is a quickly oscillating solution of system* (O) *then either*

(3) $$\lim_{t\to\infty} \|x(t)\| = 0,$$

or

$$\lim_{t\to\infty} \|x(t)\| = \infty. \tag{4}$$

Corollary 1. *If F is continuous at the origin and if there exists a bounded quickly oscillating solution then $F(0)=0$.*

Theorem 2. *If F satisfies a linear growth condition*

$$\|F(x)\| \leqq C(1 + \|x\|), \tag{5}$$

for some constant $C>0$, then every quickly oscillating solution of system (O) *satisfies* (3).

Remark. Theorem 2 generalizes both the results of A. Lasota [1], who considered equation (S), and also the results of M. Łuczyński [4], who analyzed equation (N).

Theorem 3. *If there exists a nontrivial bounded quickly oscillating solution of system* (O) *then F is not Lipschitz at the origin.*

Remark. From Theorem 3 and the Weierstrass approximation theorem, it follows that the existence of a bounded nontrivial quickly oscillating solution is not preserved under small perturbations. More exactly, if F is continuous and (O) admits a nontrivial bounded quickly oscillating solution then for every $\epsilon>0$ there exists a continuous function $G:R^n\to R^n$ with $\|G(x)\|<\epsilon$ and $G(0)=0$ such that the perturbed system $x'=F(x)+G(x)$ has no nontrivial bounded quickly oscillating solutions.

Now let us observe that if x is a C^{n-1} quickly oscillating function then by the mean value theorem, we have that $(x, x', x'', \cdots, x^{(n-1)})$ is a quickly oscillating n-vector function. Hence the properties of system (O) imply analogous properties of equation (N). Namely, we have the following corollary.

Corollary 2. *Every quickly oscillating solution x of equation* (N) *satisfies either*

$$\lim_{t\to\infty} (|x(t)| + |x'(t)| + \cdots + |x^{(n-1)}(t)|) = \infty, \tag{6}$$

or

$$\lim_{t\to\infty} (|x(t)| + |x'(t)| + \cdots + |x^{(n-1)}(t)|) = 0. \tag{7}$$

Moreover, if f satisfies a linear growth condition

$$|f(x_0, x_1, \cdots, x_{n-1})| \leqq C\left(1 + \sum_{i=0}^{n-1} |x_i|\right)$$

then only (7) *is possible. If f is continuous at the origin and $f(0, 0, \cdots, 0) \neq 0$, or if f is Lipschitz at the origin then only* (6) *is possible* (*in this case we assume x is nontrivial*).

For the second order differential equation we can replace (6) and (7) by the following theorem.

THEOREM 4. *If x is a quickly oscillating solution of equation* (S) *then either*

$$\lim_{t\to\infty} (|x(t)| + |x'(t)|) = 0, \tag{8}$$

or

$$\limsup_{t\to\infty} (x(t), -x(t), x'(t), -x'(t), x''(t), -x''(t)) = +\infty.$$

REMARK. Theorem 4 generalizes a result of Lasota and Yorke [**3**] who proved that if $x(t)$ is a quickly oscillating solution of the second order differential equation (S) then either $\limsup_{t\to\infty} |x(t)| = \infty$, or $\lim_{t\to\infty} |x(t)| = 0$. A counterexample given in [**3**] shows that Theorem 4 cannot be easily extended to the differential equations of the third order. Namely, it is possible to construct an example of the equation $x''' = h(x, x', x'')$ with continuous right-hand side and such that a solution $x(t)$ is quickly oscillating, bounded, and does not converge to zero.

3. **Proofs.** We assume throughout that the functions F, f and g are locally bounded.

LEMMA 1. *For each $M > 0$ there exists a positive $\delta = \delta(M)$ such that every solution of system* (O) *satisfying $\|x(t_0)\| \leqq M$ satisfies $\|x(t)\| \leqq 2M$ for $t \in [t_0, t_0 + \delta]$.*

The proof follows from standard arguments of differential inequalities and is omitted.

PROOF OF THEOREM 1. Let x be a quickly oscillating solution of (O) and assume (4) does not hold. Then there exists an $M > 0$ and a sequence of points $\{s_i\}$ where $s_i \to \infty$ such that $|x(s_i)| \leqq M$ for all i. For any $\epsilon > 0$, define $\delta_0 = \min(\delta(M), \epsilon/nK)$, where $K = \sup\{\|F(y)\| : \|y\| \leqq 2M\}$ and $\delta(M)$ is defined in Lemma 1. Since x is quickly oscillating there exists $T_0 = T_0(\delta_0)$ such that every coordinate x_j of x has a zero in the interval $[\bar{t}, \bar{t} + \delta_0]$ for any $\bar{t} \geqq T_0$. There exists $t_0 > T_0$ such that $\|x(t_0)\| \leqq M$, and hence from Lemma 1 we have that $\|x(t)\| \leqq 2M$ for $|t - t_0| < \delta_0$. Since x_j vanishes at some point $t_j \in [t_0, t_0 + \delta_0]$ we have, for $t \in [t_0, t_0 + \delta_0]$,

$$\begin{aligned} |x_j(t)| &= |x_j(t) - x_j(t_j)| \leqq \delta_0 \sup |x_j'(\mu)| \\ &\leqq \delta_0 \sup \|F(x(\mu))\| \leqq \delta_0 K; \end{aligned}$$

where in each case the sup is taken over the interval $[t_0, t_0+\delta_0]$; consequently, $\|x(t)\| \leqq \delta_0 Kn \leqq \epsilon$.

We may assume without loss of generality that $\epsilon \leqq M$. Since we now have $\|x(t_0+\delta_0)\| \leqq M$, then using the same method we conclude that $\|x(t)\| \leqq \epsilon$ for $t \in [t_0+\delta_0, t_0+2\delta_0]$. Continuing this process we obtain that $\|x(t)\| \leqq \epsilon$ for all $t \geqq t_0$. Since ϵ is arbitrary we have $x(t)$ satisfies (3).

Proof of Corollary 1. It is sufficient to show that $F_j(0) = 0$ for $j = 1, \cdots, n$, where $F = (F_1, \cdots, F_n)$. Since $x(t)$ is a bounded quickly oscillating solution then the $\lim_{t\to\infty} x(t) = 0$; hence $\lim_{t\to\infty} x_j(t) = 0$ for every coordinate $x_j(t)$. Since $x_j(t)$ is quickly oscillating we have the existence of a sequence of points $\{v_i\}$ such that $v_i \to \infty$ and $x_j'(v_i) = 0$. Hence, we have for all v_i, $0 = x_j'(v_i) = F_j(x(v_i))$; and since F_j is continuous at zero, we have

$$0 = \lim_{i\to\infty} F_j(x(v_i)) = F_j(0).$$

Proof of Theorem 2. Let x be a quickly oscillating solution of (O). Choose an arbitrary $\epsilon > 0$. There exists a $T = T(\epsilon)$ such that every coordinate x_j has a zero in any interval $[r, r+\epsilon]$ with $r \geqq T$. We have, for $t \in [r, r+\epsilon]$,

$$|x_j(t)| \leqq \epsilon \sup |x_j'(s)| \leqq \epsilon \sup \|x'(s)\|;$$

and consequently

$$\|x(t)\| \leqq \epsilon n \sup \|x'(s)\| \leqq \epsilon n C(1 + \sup \|x(s)\|).$$

Therefore, we have $\sup\|x(s)\| \leqq \epsilon n C(1 + \sup\|x(s)\|)$ or

$$\|x(t)\| \leqq \sup \|x(s)\| \leqq \epsilon n C/(1 - \epsilon n C), \tag{9}$$

where in each case the sup is taken over the interval $[r, r+\epsilon]$. Without any loss of generality we may assume $\epsilon n C < 1$. Using the same techniques we can show that (9) is true for $t \in [r+k\epsilon, r+(k+1)\epsilon]$ where k is any positive integer. Hence inequality (9) holds for all $t \geqq T$. Since ϵ is arbitrary, the proof is complete.

Proof of Theorem 3. Assume that F is Lipschitz at the origin; that is,

$$\|F(x) - F(0)\| \leqq L\|x\| \quad \text{for } \|x\| \leqq r,$$

where $r > 0$ and $L > 0$, and that $x(t)$ is a bounded quickly oscillating solution of (O). By Corollary 1 we have $F(0) = 0$ and consequently

$$\|F(x)\| \leqq L\|x\| \quad \text{for } \|x\| \leqq r. \tag{10}$$

According to Theorem 1 we may assume $\|x(t)\| \leqq r$ for sufficiently large t (say $t \geqq T$). Choose a positive number δ such that $\delta < \pi/2L$. Since x is quickly oscillating there exists an interval $[t_0, t_0+\delta]$ with $t_0 \geqq T$ such that every coordinate x_j of x vanishes at a point $t_j \in [t_0, t_0+\delta]$. Therefore, using (10) we can write

$$\|x'(t)\| \leqq L\|x(t)\|, \qquad t_0 \leqq t \leqq t_0 + \delta,$$

$$x_j(t_j) = 0, \qquad |t_i - t_j| \leqq \delta, \qquad i, j = 1, \cdots, n.$$

Since $\delta L < \pi/2$, we have, using a result of Lasota and Olech [2], that $x(t) \equiv 0$ in the interval $[t_0, t_0+\delta]$. By the uniqueness of the Cauchy problem (F is Lipschitzean at the origin) we conclude $x(t) \equiv 0$ in its domain.

Since Corollary 2 is a special case of the preceding results we omit the proof.

Proof of Theorem 4. Assume that x is a quickly oscillating solution of equation (S) satisfying for large t one of the following conditions:

(a) $x(t) \geqq -M$,

(b) $x'(t) \geqq -M$, or

(c) $x''(t) \geqq -M$,

where M is a positive constant. According to Corollary 2 in order to prove (8) it is sufficient to find a sequence of points $\{s_i\}$, $s_i \to \infty$ such that

$$|x(s_i)| \leqq M \quad \text{and} \quad |x'(s_i)| \leqq M. \tag{11}$$

In case (a) we observe that in each interval $[t_i, t_{i+2}]$ which contains three different zeros t_i, t_{i+1}, t_{i+2} of $x(t)$, there exists a point s_i such that $x'(s_i) = 0$ and $x(s_i) \leqq 0$. Therefore, condition (11) is satisfied. In case (b) we observe that in each interval $[t_i, t_{i+1}]$ there exists a point s_i such that $x(s_i) = 0$ and $x'(s_i) \leqq 0$. Once again (11) is satisfied. If (c) is satisfied we observe that in each interval $[t_i, t_{i+2}]$ either there exists a point u_i such that $x(u_i) = 0$ and $x'(u_i) = 0$ or there exists two points σ_i, τ_i ($\sigma_i < \tau_i$) such that $x(\tau_i) = x(\sigma_i) = 0$ and $x(t) \geqq 0$ for $t \in [\sigma_i, \tau_i]$. For this last situation, since $x' \geqq -M$ it follows from the mean value theorem

$$|x'(t)| \leqq M(\tau_i - \sigma_i) \quad \text{for } \sigma_i \leqq t \leqq \tau_i,$$

and hence

$$x(t) \leqq M(\tau_i - \sigma_i)^2 \leqq M(t_{i+2} - t_i)^2.$$

For sufficiently large i we have $(t_{i+2}-t_i)^2 \leqq 1$. Consequently, at a point v_i where x attains its maximum in the interval $[\sigma_i, \tau_i]$ we have $x'(v_i)=0$ and $0 \leqq x(v_i) \leqq M$. Setting either $s_i=u_i$ or $s_i=v_i$, respectively, we obtain (11). Therefore, every one of the conditions (a), (b) or (c) implies (11).

If either x, x', or x'' is assumed to be bounded from above, then using a similar proof we may conclude once again (11) is satisfied.

4. **Examples and concluding remarks.** If we assume that equation (S) defines a dynamical system in the plane, then we can make several remarks about quickly oscillating solutions. With the use of Corollary 2, one can show that if all solutions of (S) are quickly oscillating then either all solutions are unbounded or all solutions approach the origin asymptotically; and hence if $g(0, 0)=0$ then the origin is either globally asymptotically stable or globally asymptotically unstable. Moreover, from Theorem 3, we have that if g is Lipschitz at the origin then the origin is globally asymptotically unstable.

We now give examples of differential equations whose solutions are quickly oscillating. Consider

$$x'' = -x' - x^{1/2n+1}, \qquad n = 1, 2, 3, 4, \cdots. \tag{12}$$

All the solutions of (12) are quickly oscillating and converge to zero. To see this we observe that any solution $x(t)$ of (12) is bounded and oscillating. Moreover, it can be shown that $|x'(t_i)| \to 0$ as $t_i \to \infty$ where $x(t_i)=0$. The solution can be proved to be quickly oscillating by comparing equation (12) with the system

$$\begin{aligned} z'' &= -z^{1/2n+1}, \qquad n = 1, 2, 3, \cdots, \\ z(0) &= 0, \\ z'(0) &= p. \end{aligned}$$

If we let $T(p)$ be the first zero of $z(t)$ then it is known that $T(p) \to 0$ as $p \to 0$. It then follows for equation (12) that $(t_{i+1}-t_i) \to 0$ as $i \to \infty$. Since $x(t)$ is bounded and quickly oscillating then it converges to zero.

In a similar manner one can show that all solutions of

$$x'' = x' - x^{2n+1}, \qquad n = 1, 2, 3, \cdots,$$

are quickly oscillating and unbounded.

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UNIVERSITY OF MISSOURI, COLUMBIA, MISSOURI 65201

INSTITUTE FOR FLUID DYNAMICS AND APPLIED MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MARYLAND 20742

JAGELLONIAN UNIVERSITY, KRAKÓW, POLAND