AN EXTENSION OF MORITA'S METRIZATION THEOREM

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ABSTRACT. Morita proved that metrizability of a T_0 space is equivalent to the existence of a sequence of locally finite closed covers which satisfies a refinement condition. We introduce the definition of a cushioned pair-semidevelopment and prove that the existence of a cushioned pair-semidevelopment in a T_0 space is equivalent to the metrizability of the space. In addition to Morita's theorem, it is seen that several other well-known metrization theorems are also immediate corollaries to the new theorem.

A topological space X is said to be *semidevelopable* [1] if there is a sequence of (not necessarily open) covers of X , $\gamma = \{\gamma_n\}_1^\infty$, such that for each $x \in X$, $\{\text{St}(x, \gamma_n)\}_1^\infty$ is a neighborhood base at x . In this case, γ is called a *semidevelopment*. If γ and δ are collections of subsets of X , then we say that γ is *cushioned* in δ if one can assign to each $G \in \gamma$ a $D(G) \in \delta$ such that, for every $\gamma' \subset \gamma$,

$$\text{Cl}(\cup\{G \mid G \in \gamma'\}) \subset \cup\{D(G) \mid G \in \gamma'\}.$$

By a *cushioned pair-semidevelopment* for X we shall mean a pair of semidevelopments (γ, δ) such that γ_n is cushioned in δ_n , for each n .

REMARKS. (i) If $\gamma = \{\gamma_n\}_1^\infty$ is a sequence of closure-preserving¹ closed covers of a space X such that for each $x \in X$ and neighborhood U of x , there is an integer n such that $\text{St}(x, \gamma_n) \subset U$, then (γ, γ) is a cushioned pair-semidevelopment.

(ii) If X is a metric space, Σ_n is the collection of all spheres of radius less than $1/n$, for $n = 1, 2, \dots$, and γ_n is a locally finite closed cover of X which refines Σ_n , then (γ, γ) is a cushioned pair-semidevelopment.

(iii) If γ and δ are open covers of X such that $\gamma^* < \delta$, then γ is cushioned in δ .²

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¹ A collection γ is said to be *closure-preserving* if, for every $\gamma' \subset \gamma$, $\text{Cl}(\cup\{G \mid G \in \gamma'\}) \subset \cup\{\text{Cl}(G) \mid G \in \gamma'\}$.

² If γ is a collection of subsets of X , then $\gamma^* = \{\text{St}(G, \gamma) \mid G \in \gamma\}$, where $\text{St}(G, \gamma) = \cup\{H \mid H \in \gamma, H \cap G \neq \emptyset\}$.

Before proceeding to the main theorem we note the following theorem which is due to Michael [6].

THEOREM 1. *A T_1 space X is paracompact if and only if every open cover of X has a σ -cushioned open refinement.*

In [6], immediately following the proof of this theorem, Michael comments that it is sufficient to assume that every open cover has a σ -cushioned refinement, $\gamma = \bigcup \{\gamma_n \mid n = 1, 2, \dots\}$, such that the sets $\text{Int}(\bigcup \{G \mid G \in \gamma_n\})$, $n = 1, 2, \dots$, cover X . The following theorem is a corollary to Theorem 1 and this comment.

THEOREM 2. *A T_1 space X is paracompact if and only if every open cover of X has a σ -cushioned refinement, $\gamma = \bigcup \{\gamma_n \mid n = 1, 2, \dots\}$, such that for each $x \in X$ there is an integer n such that $x \in \text{Int St}(x, \gamma_n)$.*

THEOREM 3. *A space is metrizable if and only if it is T_0 and has a cushioned pair-semidevelopment.*

PROOF. The "only if" part of the theorem follows from remark (ii) and the fact that every metrizable space is paracompact.

For the other half of the theorem, assume that (γ, δ) is a cushioned pair-semidevelopment. We can assume that $\gamma_{n+1} < \gamma_n$ for each n . Because if not, we can replace γ_n with $\gamma'_n = \gamma_1 \wedge \gamma_2 \wedge \dots \wedge \gamma_n$ for each n . Then $\gamma' = \{\gamma'_n\}_1^\infty$ is easily seen to be a semidevelopment and since $\gamma'_n < \gamma_n$ for each n , each γ'_n is cushioned in δ_n . For $G \in \gamma_n$, let $D(G) \in \delta_n$ be chosen so that the condition of γ_n being cushioned in δ_n is satisfied.

That X is paracompact follows from Theorem 2, since a T_0 semi-developable space is a T_1 space. We shall complete the proof by exhibiting a σ -locally finite basis for X and then applying the Nagata-Smirnov Theorem [9], [10]. Let $\Sigma_0 = \{\{x\} \mid x \in X \text{ and } x \text{ is an isolated point}\}$. For $n > 1$, let Σ_n be a locally finite open refinement of $\lambda_n = \{\text{Int St}(x, \gamma_n) \mid x \in X\}$.

Also, if we let

$$\Sigma_{0,n} = \{\{x\} \mid x \in X \text{ and } x = \text{St}(x, \delta_n)\},$$

then $\Sigma_{0,n}$ is locally finite, and $\Sigma_0 = \bigcup \{\Sigma_{0,n} \mid n = 1, 2, \dots\}$. Hence $\Sigma = \bigcup \{\Sigma_n \mid n = 1, 2, \dots\} \cup \Sigma_0$ will satisfy the Nagata-Smirnov condition, if we can show that it is a basis for X .

First define $W_n(x) = X \setminus \bigcup \{G \mid G \in \gamma_n, x \notin D(G)\}$ for each $x \in X$ and each positive integer n . Then, since γ_n is cushioned in δ_n , we have that $x \notin \bigcup \{D(G) \mid G \in \gamma_n, x \notin D(G)\} \supset \text{Cl}(\bigcup \{G \mid G \in \gamma_n, x \notin D(G)\}) \supset X \setminus W_n(x)$, and therefore, $W_n(x)$ is a neighborhood of x . Moreover, $\text{St}(W_n(x), \gamma_n) \subset \text{St}(x, \delta_n)$ for each x and n , since if $G \in \gamma_n$ and

$G \cap W_n(x) \neq \emptyset$, then $x \in D(G)$, by definition of $W_n(x)$, and hence $G \subset D(G) \subset \text{St}(x, \delta_n)$.

Now assume that U is a neighborhood of x , where x is not an isolated point of X . Then there is an integer k such that $\text{St}(x, \delta_k) \subset U$, and there is an integer n such that $\text{St}(x, \delta_n) \subset W_k(x)$. In fact, we can choose $n > k$, since X is T_1 and x is not isolated. Then, since Σ_n is a cover of X , there is $S \in \Sigma_n$ such that $x \in S$. Then there is $z \in X$ such that $S \subset \text{St}(z, \gamma_n)$, since $\Sigma_n < \lambda_n$. Thus $x \in \text{St}(z, \gamma_n)$, and therefore, $z \in \text{St}(x, \gamma_n) \subset \text{St}(x, \delta_n) \subset W_k(x)$. It follows from the comment above that $\text{St}(z, \gamma_k) \subset \text{St}(W_k(x), \gamma_k) \subset \text{St}(x, \delta_k) \subset U$. Hence $x \in S \subset \text{St}(z, \gamma_n) \subset \text{St}(z, \gamma_k) \subset U$, since $\gamma_n < \gamma_k$. It follows that Σ is a basis for X .

COROLLARY. *A space X is metrizable if and only if it is T_0 and there is a sequence of closure-preserving closed covers of X , $\gamma = \{\gamma_n\}_1^\infty$, such that for each $x \in X$ and each neighborhood U of x , there is an integer n such that $\text{St}(x, \gamma_n) \subset U$.³*

PROOF. Remark (i) and Theorem 3.

The following theorem is due to Morita [8] and follows immediately from the corollary to Theorem 3. It should be noted that Theorem 3 can also be proved using Morita's theorem with techniques similar to those in the proof above. However, since Morita's theorem follows trivially from Theorem 3, one can consider Theorem 3 to be an extension of Morita's theorem.

THEOREM 4. *A space X is metrizable if and only if it is T_0 and there is a sequence of locally finite closed covers, $\gamma = \{\gamma_n\}_1^\infty$, such that for each $x \in X$ and each neighborhood U of x , there is an integer n such that $\text{St}(x, \gamma_n) \subset U$.*

Theorem 3 may be regarded as a metrization theorem for semi-stratifiable spaces [4], since they are precisely the spaces which have a σ -cushioned pair-network (= σ -biconservative pair-network) [5].⁴ Also it may be applied to semimetric spaces, since they are the T_0 semidevelopable spaces [1].

Incorporated in the next theorem are the statements of several well-known metrization theorems. We will show that in each case the

³ The proof of the necessity of the condition for metrizability in the Corollary is almost obvious and consequently will be omitted. This will also be done for the remaining theorems.

⁴ A collection, γ , of ordered pairs of subsets of a space X is a σ -cushioned pair-network if $\gamma = \bigcup \{\gamma_n \mid n = 1, 2, \dots\}$, where $\gamma_n = \{(G_1^\alpha, G_2^\alpha) \mid \alpha \in A_n\}$, and (i) for any $x \in X$ and any neighborhood U of x , there is an integer n and an $\alpha \in A_n$ such that $x \in G_1^\alpha \subset G_2^\alpha \subset U$ and (ii) $\text{cl}(\bigcup \{G_1^\alpha \mid \alpha \in B_n\}) \subset \bigcup \{G_2^\alpha \mid \alpha \in B_n\}$, for each n and any $B_n \subset A_n$.

sufficiency of the condition for metrizability follows easily from Theorem 3.

THEOREM 5. *The following conditions are each necessary and sufficient for the metrizability of a space X .*

(1) (ALEXANDROFF-URYSOHN [2]). X is T_0 and there is a sequence of open covers, $\{\lambda_n\}_1^\infty$, such that $\lambda_{n+1}^* < \lambda_n$ for each n , and for each $x \in X$, $\{\text{St}(x, \lambda_n)\}_1^\infty$ is a neighborhood base at x .

(2) (STONE-ARHANGEL'SKIĬ [11], [3]). X is T_0 and there is a sequence of open covers, $\{\lambda_n\}_1^\infty$, such that for each $x \in X$ and each neighborhood U of x , there is a neighborhood V of x and an integer n such that $\text{St}(V, \lambda_n) \subset U$.

(3) (MORITA [7]). X is T_0 and there is a sequence of open covers, $\{\lambda_n\}_1^\infty$, such that for each $x \in X$, $\{\text{St}^2(x, \lambda_n)\}_1^\infty$ is a neighborhood base at x .⁵

PROOF. (1) For each n , let $\gamma_n = \lambda_{2n}$ and $\delta_n = \lambda_{2n-1}$. In order to see that (γ, δ) is a cushioned pair-semidevelopment, we note that γ_n is cushioned in δ_n for each n , by remark (iii).

(2) Without loss of generality we may assume $\lambda_{n+1} < \lambda_n^*$ for each n . Let $\gamma_n = \lambda_n$ and $\delta_n = \lambda_n^*$. By applying the property of $\{\lambda_n\}_1^\infty$ twice one sees that δ is a semidevelopment. It then follows that (γ, δ) is a cushioned pair-semidevelopment.

(3) See proof of (2).

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⁵ If λ is a collection of subsets of X , $x \in X$, and n is a positive integer, we have the following recursive definition: $\text{St}^n(x, \lambda) = \text{St}(\text{St}^{n-1}(x, \lambda), \lambda)$.

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