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RECONSTRUCTING GRAPHS[1]

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ABSTRACT. Every graph G determines a collection M of maximal vertex proper subgraphs $G_i = G - v_i$ and a collection M' of maximal edge proper subgraphs $G^i = G - e_i$. In this paper we prove that a graph G, on at least three edges and without isolated vertices, can be reconstructed, up to isomorphism, from the collection M' if it can be reconstructed, up to isomorphism, from the collection M.

Let G be a graph, v a vertex of G and e an edge of G. We will denote by G_v the subgraph of G obtained by deleting v and all edges incident to v, and G^e the subgraph of G obtained by deleting e. The following problem was proposed by Ulam [5].

THE VERTEX PROBLEM. If G and H are graphs, $|V(G)| > 2$, and σ: $V(G) \to V(H)$ is a one-to-one correspondence such that $G_v \simeq H_{\sigma(v)}$ for all $v \in V(G)$, then $G \simeq H$.

A corresponding problem was proposed by Harary in [2].

THE EDGE PROBLEM. If G and H are graphs, $|E(G)| > 3$, and σ: $E(G) \to E(H)$ is a one-to-one correspondence such that $G^e \simeq H^{\sigma(e)}$ for all $e \in E(G)$, then $G \simeq H$.

Hemminger in [4] has shown:

LEMMA 1. *The edge problem is true for a graph G, (i.e. for all H as in the edge problem we have $G \simeq H$), if and only if the vertex problem is true for the line graph, $L(G)$, of G.*

It is the purpose of this paper to show:

THEOREM 1. *If the vertex problem is true for a graph G, which contains no isolated vertices, and $|E(G)| > 3$, then the edge problem is true for G.*

All terms referred to in this paper can be found in [3].

Before proving Theorem 1 we will need to recall two results which appear in [1].

LEMMA 2. *If the conditions of the edge problem are satisfied, then G and H have the same degree sequence.*

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LEMMA 3. *If the conditions of the edge problem are satisfied, then every type of edge proper subgraph which occurs in G or H occurs the same number of times in both.*

PROOF OF THEOREM 1. For any graph K define a partition of $V(K)$ by the equivalence relation $v\,R\,w$ if and only if $K_v \simeq K_w$. Denote by $R_v(K)$ the equivalence class containing v. Let $D_n(K) = \{v \in V(K): dg(v) = n\}$.

Now suppose G and H satisfy the conditions of the edge problem. By Lemma 2 we know that for any integer n, $|D_n(G)| = |D_n(H)|$. So we can let $k = \min\{n: D_n(G) \neq \emptyset\} = \min\{n: D_n(H) \neq \emptyset\}$. By Lemma 3 we know that for any v in $D_k(G)$ we have a w in $D_k(H)$ such that $G_v \simeq H_w$ and $|R_v(G)| = |R_w(H)|$. And so we have a one-to-one correspondence $\tau: D_k(G) \to D_k(H)$ such that $G_v \simeq H_{\tau(v)}$ for all v in $D_k(G)$.

Now consider $D_{k+1}(G)$ and any v in $D_{k+1}(G)$. By Lemma 3 there are the same number of subgraphs of H isomorphic to G_v as there are in G. However now there are two ways to obtain such a subgraph. One possible way is to remove a vertex of $D_k(H)$ and an edge not incident to it. But from the result of the preceding paragraph there are the same number of ways to do this in G as in H. The only other way is by removing a vertex of $D_{k+1}(H)$. So there must be a w in $D_{k+1}(H)$ such that $G_v \simeq H_w$ and $|R_v(G)| = |R_w(H)|$. And so we have a one-to-one correspondence $\tau: D_{k+1}(G) \to D_{k+1}(H)$ such that $G_v \simeq H_{\tau(v)}$ for all v in $D_{k+1}(G)$.

Similarly for any $n \geqq k$ we can show the existence of a one-to-one correspondence $\tau: D_n(G) \to D_n(H)$ such that $G_v \simeq H_{\tau(v)}$ for all v in $D_n(G)$. Since for some m we have $\bigcup_{n=1}^{m} D_n(G) = V(G)$, and $D_i(G) \cap D_j(G) = \emptyset$ whenever $i \neq j$, we have a one-to-one correspondence $\tau: V(G) \to V(H)$ such that $G_v \simeq H_{\tau(v)}$ for all v in $V(G)$. And so, since the vertex problem is true for G, we have $G \simeq H$.

As a consequence of Theorem 1 and Lemma 1 we have:

THEOREM 2. *If the vertex problem is true for a connected graph G, then the vertex and edge problems are true for* $L^n(G)$, $n \geqq 0$.

THEOREM 3. *If the edge problem is true for a connected graph G, then the vertex and edge problems are true for* $L^n(G)$, $n \geqq 1$.

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