ON THE DIRECT PRODUCT OF V -GROUPS

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ABSTRACT. Let G and H be ordered groups such that every full order on a subgroup extends to a full order on the group; then the direct product, $G \times H$, need not have this property. In fact a stronger result holds.

A group is called a V -group if every full order on a subgroup may be extended to a full order on the group. Kargapolov [2] showed that an arbitrary torsion-free group G is a V -group if and only if G has an abelian normal subgroup B such that (1) the factor group G/B is abelian, and (2) for arbitrary β in B and α in $G - B$ there are positive integers $m \neq n$ such that $\alpha^{-1}\beta^m\alpha = \beta^n$.¹ In particular, torsion-free abelian groups are V -groups. The necessity of Kargapolov's theorem is essentially due to Terehov [6].

LEMMA. Let G have the presentation $G = \{a, b \mid ba = ab^p\}$, where p is a fixed prime. Then G is a V -group.

PROOF. First of all, G is torsion-free since it admits a full order [1] (direct verification is not difficult). Next, we construct a subgroup B of G satisfying the conditions of Kargapolov's theorem.

By definition of G , $b^p = a^{-1}ba$, and so by induction

$$b^{p^k} = a^{-k}ba^k, \quad \text{all } k \geq 0.$$

Moreover $(a^kba^{-k})^{p^k} = a^k(b^{p^k})a^{-k} = b$, so that we may define

$$b^{p^{-k}} \equiv a^kba^{-k}, \quad \text{all } k \geq 0.$$

It follows that

$$(*) \quad b^{p^i} = a^{-i}ba^i \quad \text{for every integer } i.$$

Consider the subgroup B of G generated by $\{b^{p^w} : w \text{ an integer}\}$. B consists of all p -adic rational powers of b , i.e., all b^d where $d = \sum_j \pm p^{w_j}$ for integers w_j , hence B is torsion-free abelian. Also B is nor-

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¹ The statement (but not the proof) of this theorem in [2] is incomplete, for it does not mention that $\beta \in B$. Further, the hypotheses cannot be weakened to allow m, n arbitrary integers (see [4, p. 226] for a counterexample).

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mal in G since $a^{-i}b^w a^i = b^{w+i} \in B$, for all integers i and w . Thus given an integer $i \neq 0$ and a p -adic rational d there is a p -adic rational $d' \neq d$ with the same sign as d such that $a^{-i}b^d a^i = b^{d'} \in B$. It follows easily from this that there are positive integers $m \neq n$ with

$$a^{-i}\beta^m a^i = \beta^n \quad \text{for all } \beta \in B.$$

Kargapolov's second condition follows from this once we note that every $g \in G - B$ may be written in the form $g = a^i \beta$, $i \neq 0$, $\beta \in B$. Q.E.D.

Finally G/B is infinite cyclic, for it is generated by aB and is infinite lest some power of a commute with b , violating the definition of G .

THEOREM. *There are V -groups G, H with $G \times H$ not a V -group.*

PROOF. Let $G = \{a, b \mid ba = ab^2\}$ and $H = \{c, d \mid dc = cd^3\}$; note that G and H are V -groups by the lemma. Were $G \times H$ a V -group, it would have an abelian normal subgroup B of the type described in Kargapolov's theorem. Since $(G \times H)/B$ is abelian, B must contain the commutator subgroup $(G \times H)' = G' \times H'$. Setting $i \equiv 1$ in (*) above shows b^2 is in G' and d^2 is in H' . So

$$(b^2, d^2) \in G' \times H' \subseteq B.$$

Now

$$(a, c)^{-1}(b^2, d^2)(a, c) = (b^4, d^6).$$

Thus, if there were positive integers $m \neq n$ such that

$$(a, c)^{-1}(b^2, d^2)^m(a, c) = (b^2, d^2)^n,$$

we would have

$$(b^{4m}, d^{6m}) = (b^{2n}, d^{2n})$$

which is impossible since G and H are torsion-free. Q.E.D.

A group G is called a *VAN-group* if every full order on every abelian normal subgroup extends to a full order on the group. G is called a *V*-group* if every *partial* order on every subgroup extends to a full order for G .

The following corollary to the previous theorem puts it in stronger terms.

COROLLARY. *The direct product of V^* -groups need not be a VAN-group.*

PROOF. We claim that a solvable VAN-group is a V^* -group. Assuming this for the moment, the corollary follows, for the group $G \times H$ in the proof of the theorem is solvable and thus, if also VAN, is V^* and hence V .

It remains to establish the claim. In fact, the classes V^* and solvable VAN coincide. A V^* -group is solvable VAN , since V^* trivially implies VAN and V^* , being V , implies solvable by Kargapolov's theorem above (note that a V -group is torsion-free since it admits a full order). Conversely, a solvable VAN -group is V^* , since a solvable VN -group (obvious meaning) is V^* (see [3]) and a solvable VAN -group is VN [5]. (Terehov [5] actually shows that a solvable V -group satisfies conditions (1) and (2) of Kargapolov's theorem, but that proof applies as well to solvable VAN -groups; thus a solvable VAN -group is V by Kargapolov's theorem, and hence VN .)

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