ON THE STABILITY OF THE FUNCTIONAL EQUATION

$$\phi[f(x)] = g(x)\phi(x) + F(x)$$

E. TURDZA

ABSTRACT. This paper extends the results of D. Brydak [1] on the stability of the functional equation in the title.

1. This paper is concerned with the stability (defined below) of the functional equation

$$(1) \quad \phi[f(x)] = g(x)\phi(x) + F(x) \quad (a < x < b)$$

where ϕ is unknown and f, g, F are given functions about which the following blanket hypotheses are made:

H1. $f, g, F \in C[a, b]$; $g(x) \neq 0$ ($a \leq x \leq b$).

H2. $f(x)$ is strictly increasing and for some $\xi, a \leq \xi \leq b$,

$$x < f(x) < \xi \quad (a < x < \xi),$$

$$\xi < f(x) < x \quad (\xi < x < b).$$

REMARK. The values $a = -\infty, b = \infty$ need not be excluded, but if $a = -\infty$ we must assume that f, g, F tend to limits as $x \rightarrow -\infty$. Similarly for $b = \infty$.

By the hypotheses H1 and H2,

$$f(\xi) = \xi \quad \text{and} \quad f([a, b]) \subset [a, b].$$

In this paper f^n denotes the composition product $f \circ f \circ \cdots \circ f$, n times. f^{-1} is the inverse function of f , $f^{-n} = (f^{-1})^n$.

Following Brydak [1] we define stability as follows.

Let $S\psi(x) = g[f^{-1}(x)]\psi[f^{-1}(x)] + F(f^{-1}(x))$ for all x belonging to a subinterval I of $[a, b]$. S is a transformation $S: C(I) \rightarrow C(f(I))$.

It is easily verified that

$$(2) \quad S^n\psi(x) = \prod_{j=1}^n g[f^{-j}(x)] \cdot \psi(f^{-n}(x)) + h_n(x),$$

where h_n is independent of ψ . If ϕ is a solution of (1) in I , then

$$S\phi[f(x)] = \phi[f(x)] \quad (x \in I)$$

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and so, by iteration,

$$S^n \phi[f^n(x)] = \phi[f^n(x)] \quad (x \in I).$$

The equation (1) is called *stable*, if there is a $K > 0$ such that for every $\psi \in C(I)$ which satisfies

$$(3) \quad \|\psi(f^n(x)) - S^n \psi(f^n(x))\| < \epsilon \quad (n = 1, 2, \dots)$$

there is a solution ϕ of (1) such that

$$\|\psi(x) - \phi(x)\| < K\epsilon \quad (x \in I).$$

Here $\|\cdot\|$ is the $C(I)$ norm.

2. D. Brydak [1] gave conditions under which (1) is stable in an interval $[\xi, x_0]$ ($x_0 < b$).

We add here

THEOREM 1. *The equation (1) is stable in every interval $[x_0, b]$ ($\xi < x_0 < b$), if either*

- (a) *for some integer r , $f^r(b) < x_0$, or*
- (b) *$g(x) \geq 1$ in a neighborhood $(x_1, b]$ of b .*

We need the following lemma, proved in [2].

LEMMA. *Let $\xi < x_0 < b$. Then, by H2,*

$$I_0 = [f(x_0), x_0] \subset [\xi, b].$$

Let $\phi_0(x) \in C(I_0)$, $\phi_0(f(x_0)) = S\phi_0(f(x_0))$ and $\prod_{i=0}^{n-1} g[f^i(x)]$ tends to zero uniformly on I_0 .

Then there exists a continuous solution ϕ of (1) in (ξ, b) such that

$$\phi(x) = \phi_0(x) \quad (x \in I_0).$$

PROOF OF THEOREM 1. Let $\psi \in C[f(x_0), b]$ and let ψ satisfy (3) where the norm is the supremum norm in $[f(x_0), b]$. Let $\eta(x)$ be a linear function which decreases from 1 to 0 as x increases from $f(x_0)$ to x_0 . Put

$$\phi_0(x) = \psi(x) + \eta(x) \{S\psi(f(x_0)) - \psi(f(x_0))\}.$$

By (3),

$$(4) \quad |\phi_0(x) - \psi(x)| < \epsilon \quad (x \in I_0)$$

and, by the definition of S ,

$$\begin{aligned} S\phi_0(f(x_0)) &= g(x_0)\phi_0(x_0) + F(x_0) = g(x_0)\psi(x_0) + F(x_0) \\ &= S\psi(f(x_0)) = \phi_0(f(x_0)). \end{aligned}$$

Applying the Lemma with this choice of ϕ_0 we obtain a solution ϕ of (1) in (ξ, b) . This solution satisfies, for any $x \in (\xi, b)$ and any natural integer,

$$\begin{aligned} & | \phi(f^k(x)) - \psi(f^k(x)) | \\ (5) \quad &= | S^k \phi(f^k(x)) - \psi(f^k(x)) | \\ &\geq | S^k \phi(f^k(x)) - S^k \psi(f^k(x)) | - | \psi(f^k(x)) - S^k \psi(f^k(x)) |. \end{aligned}$$

For every $x \in [x_0, b)$, there is a unique natural integer k such that $f^k(x) \in I_0$. (Otherwise, by H2, $f^k(x) \rightarrow X$, $x_0 \leq X$ as $k \rightarrow \infty$. But then $f(X) = X$, contradicting H2.)

Choosing k such that $f^k(x) \in I_0$ we have, by (4), (5), (2) and (3),

$$\begin{aligned} (6) \quad \epsilon &> | \phi_0(f^k(x)) - \psi(f^k(x)) | = | \phi(f^k(x)) - \psi(f^k(x)) | \\ &> G_k(x) | \phi(x) - \psi(x) | - \epsilon, \end{aligned}$$

where $G_k(x) = \prod_{j=0}^{k-1} g(f^j(x))$.

The theorem follows from (6), if we can prove the existence of a positive constant m such that

$$| G_k(x) | \geq m > 0 \quad (k = 1, 2, \dots; x \in [x_0, b]).$$

In case (a), $k \leq r$ and we can choose

$$m = \inf_{0 \leq k \leq r} \inf_{x_0 \leq x \leq b} | G_k(x) |.$$

In case (b), let M be the integer such that $f^M(x_1) \in I_0$. For any x in $[x_0, x_1]$ we can use the argument of case (a) with $r = M$. For $x_1 < x < b$ there is an index l such that $y = f^l(x) < x_1 \leq f^{l-1}(x)$. By case (a) and (6) with $k = l$ there is a constant m such that

$$\begin{aligned} 2\epsilon/m &> | \phi(y) - \psi(y) | = | \phi(f^l(x)) - \psi(f^l(x)) | \\ &> G_l(x) | \phi(x) - \psi(x) | - \epsilon > | \phi(x) - \psi(x) | - \epsilon, \end{aligned}$$

since every factor in $G_l(x)$ is ≥ 1 by hypothesis.

REFERENCES

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KRAKÓW, UL. IDZIKOWSKIEGO 19/4, POLAND