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# SPLITTINGS OF HOCHSCHILD'S COMPLEX FOR COMMUTATIVE ALGEBRAS

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ABSTRACT. Barr has shown that one may split Hochschild's complex for commutative algebras into Harrison's complex plus a shuffle subcomplex when working over a field of characteristic zero. We construct a splitting here for the above complex over a ring containing a field which does not have characteristic two and this splitting has Barr's splitting as a special case.

1. **Introduction.** In [1], Barr noted that Harrison's homology could be regarded as a direct summand of Hochschild's homology when working over a field, $k$, of characteristic zero. In order to split Hochschild's complex, Barr constructed an idempotent in $k\Sigma_n$ for all $n \geqq 1$ and showed that this idempotent was a chain map which had for its kernel the "shuffle" subcomplex. The purpose of this paper is to generalize this splitting to commutative algebras over rings containing fields of any characteristic not equal to two.

2. **The complex, shuffles and representations.** In [1], it is shown that, if one considers a commutative algebra, $A$, over an arbitrary commutative ring, $k$, and then takes coefficients only in symmetric $A$-modules, Hochschild's complex in the $n$th dimension is just $C_nA = A \otimes A^{(n)}$. The $n$th tensor power of $A$ is denoted by $A^{(n)}$ and tensor products are taken over $k$ unless otherwise specified. Symmetric $A$-modules are known to be the same as left $A$-modules (see [1]). Then the map $d_n: C_nA \to C_{n-1}A$ by

$$d_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = a_0a_1 \otimes \cdots \otimes a_n - a_0 \otimes a_1a_2 \otimes \cdots \otimes a_n + \cdots + (-1)^n a_0a_n \otimes a_1 \otimes \cdots \otimes a_{n-1}$$

will be $A$-linear and a boundary operator. We will denote the entire complex just defined by $C_*A$, and, in agreement with the notation of other authors, we denote an element of $C_nA$ by $a_0[a_1, \cdots, a_n]$. Let

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us note, for future reference, that $d_1: C_1A \to C_0A$ is zero. From the foregoing we may now conclude that

$$\text{Hoch}_* (A, M) = H_*(C_*A \otimes_A M) \quad \text{and}$$
$$\text{Hoch}^* (A, M) = H^*(\text{Hom}_A (C_*A, M)).$$

Now let $\Sigma_n$ denote the symmetric group on $n$-letters and define an action of $\Sigma_n$ on $C_nA$ by

$$\pi^{-1}(a_0[a_1, \cdots, a_n]) = a_0[a_{\pi(1)}, \cdots, a_{\pi(n)}].$$

Thus $C_nA$ becomes a $k\Sigma_n$-module. We shall define a shuffle, $s_{i,n-i}$, $0 \leqq i \leqq n$, in $k\Sigma_n$ by $s_{0,n} = s_{n,0} = 1$ and

$$s_{i,n-i}(a_0[a_1, \cdots, a_n]) = a_0[a_1] \otimes s_{i-1,n-i}([a_2, \cdots, a_n]) + (-1)^i a_0[a_{i+1}] \otimes s_{i,n-i-1}([a_1, \cdots, a_i, a_{i+2}, \cdots, a_n]).$$

Then we have the following proposition whose proof appears partly in [**1**] and partly in [**3**].

2.1. Proposition.

$$\begin{aligned} d_n s_{i,n-i}&(a_0[a_1, \cdots, a_n]) \\ &= s_{i-1,n-i}(d_i a_0[a_1, \cdots, a_i] \otimes [a_{i+1}, \cdots, a_n]) \\ &\quad + (-1)^i s_{i,n-i-1}(a_0[a_1, \cdots, a_i] \otimes d_{n-2}[a_{i+1}, \cdots, a_n]). \end{aligned}$$

Because of the above, one may consider the shuffles as multiplication in the differential graded algebra $C_*A$. The complex $C_*A$, has an augmentation, i.e., a map of complexes to $A$ which is considered as a trivial complex over itself. The kernel of this mapping is a subcomplex of $C_*A$ and we will call it $J_*A$. Since we noted before that $d_1$ was zero, it is easy to see that $J_nA = C_nA$ if $n > 0$ and $J_0A = 0$. Now consider $J_*^2A$ which we define to be that subcomplex of $J_*A$ generated by nontrivial shuffles. We now set $Ch_*A = J_*A / J_*^2A$. Then the differential and grading of $J_*A$ induce a differential and grading on $Ch_*A$. We define the $n$th Harrison homology and cohomology groups of $A$ with coefficients in the left $A$-module $M$ to be

$$\text{Harr}_n(A, M) = H_n(Ch_*A \otimes_A M) \quad \text{and}$$
$$\text{Harr}^n(A, M) = H^n(\text{Hom}_A(Ch_*A, M))$$

and we denote the total homology and cohomology by $\text{Harr}_*(A, M)$ and $\text{Harr}^*(A, M)$ respectively.

Of special importance to us in the ring $k\Sigma_n$ will be the element $E_n$ defined in the following manner. Let $\text{sgn}: \Sigma_n \to k$ be the group homomorphism sending elements of the alternating subgroup to 1 and other elements to $-1$. Then we may extend sgn to a ring homomorphism also called $\mathrm{sgn}: k\Sigma_n \to k$. Let $E_n = \sum_{\pi \in \Sigma_n} (\mathrm{sgn}(\pi)) \cdot \pi$. If $u \in k\Sigma_n$, then, clearly, $u \cdot E_n = \mathrm{sgn}(u) \cdot E_n$.

2.2. LEMMA (BARR [1]). *Let $a_0[a_1, \cdots, a_n] \in J_nA$. Then $d_nE_n(a_0[a_1, \cdots, a_n]) = 0$. Furthermore, if $u \in k\Sigma_n$ and*

$$d_nu(a_0[a_1, \cdots, a_n]) = 0$$

*for all $a_0[a_1, \cdots, a_n] \in J_nA$, and arbitrary $A$, then $u$ is some multiple of $E_n$.*

3. **The splittings.** We are interested in splitting Hochschild's complex. Barr has shown that, if one works over a field of characteristic zero, then the complex can be split in such a way that Harrison's groups are direct summands of Hochschild's. We shall use techniques which will give us Barr's theorem as a special case of a more general theorem.

Earlier, we noted that each $s_{i,n-i}$ could be considered as an element of $k\Sigma_n$. We now define another element, $s_n$, of $k\Sigma_n$ in the following way. If $n=1$, $s_1=0$; if $n \geqq 2$, $s_n = \sum_{i=1}^{n-1} s_{i,n-i}$. It is clear that $s_{i,n-i}: J_nA \to J_nA$ need not be a chain map, but Barr in [1] proves 3.1, 3.2 and 3.3.

3.1. LEMMA. $d_ns_n = s_{n-1}d_n$.

3.2. LEMMA. $\mathrm{sgn}(s_{i,n-i}) = \binom{n}{i}$.

3.3. COROLLARY. $\mathrm{sgn}(s_n) = 2^n - 2$.

3.4. PROPOSITION. *$((2^n-2)-s_n) \cdots (2-s_n)s_{i,n-i} = 0$ for all $1 \leqq i \leqq n$ and all $n \geqq 1$.*

PROOF. We proceed by induction, the case for $n=1$ being trivial. Now assume the proposition is true for $n-1$. Then

$$\begin{aligned} d_n((2^{n-1} - 2) - s_n) \cdots (2 - s_n)s_{i,n-i} \\ = ((2^{n-1} - 2) - s_n) \cdots (2 - s_n)(s_{i-1,n-i}(d_i \otimes 1) \\ + (-1)^i s_{i,n-i-1}(1 \otimes d_{n-i})) \end{aligned}$$

by 2.1 and 3.1. By induction, both terms in the above sum are zero. This implies $((2^{n-1}-2)-s_n) \cdots (2-s_n)s_{i,n-i}$ is some multiple, say $r$, of $E_n$. Thus

$$\begin{aligned} ((2^n - 2) - s_n) \cdots (2 - s_n)s_{i,n-i} &= ((2^n - 2) - s_n) \cdot r \cdot E_n \\ &= ((2^n - 2) - \mathrm{sgn}(s_n)) \cdot r \cdot E_n = 0. \end{aligned}$$

Now suppose we consider $e_n' = ((2^n-2)-s_n)\cdots(2-s_n)\in k\Sigma_n$ where $k$ is a field of characteristic $p$. Now consider $(e_n')^2$. If we expand $(e_n')^2$ in terms of $s_n$, then every term, excepting only the first, is a multiple of $e_ns_n$ and is zero by Proposition 3.4. Thus $(e_n')^2 = \{\prod_{2\le i\le n}(2^i-2)\}e_n'$. If we could multiply $e_n'$ by the inverse of $\prod_{2\le i\le n}(2^i-2)$ we could make $e_n'$ into an idempotent. Unhappily, this is not always possible since that product might be zero in $k$. Certainly, it is possible when we are working over a field of characteristic zero. Furthermore, if we are working with a field of characteristic $p$, and 2 is a primitive root modulo $p$, then we may divide by the above product in dimensions up to but not including $p$. In order to investigate this further, we shall need some facts about idempotents in arbitrary rings.

3.5. PROPOSITION. *Let $T$ be a (possibly noncommutative) ring. Let $a$ be a nonnilpotent element of $T$ such that $a^2-a$ is nilpotent and let $m$ be the least integer with $(a^2-a)^m=0$. Then there is a nonzero polynomial, $p_m(x)$, with integral coefficients and $a^m\{p_m(a)\}^m$ is a nonzero idempotent.*

3.6. PROPOSITION. $p_m(x)=1+(1-x)+\cdots+(1-x)^{m-1}$.

The proof of 3.6 is an easy (but messy) induction, so we omit it. For 3.5 we refer the reader to Herstein [**6**, p. 22]. We note, for future reference, that Proposition 3.5 implies $a^m=a^{m+1}p_m(a)$. Let us now return to our consideration of $J_*A$. Let $j$ be the order of two in the group of units modulo $p$. Let $r$ be the inverse of $\prod_{2\le i\le j}(2^i-2)$ in that group of units. Let us now set

$$w_n = r(((2^j-2)-s_n)\cdots(2-s_n)).$$

Then $w_n$ will be a polynomial in $s_n$ with constant coefficient 1.

3.7. PROPOSITION. *$w_n^2-w_n$ is nilpotent.*

PROOF. In the ring $Z\Sigma_n$, we have the equation

$$(*)\qquad ((2^n-2)-s_n)\cdots(2-s_n)s_{i,j}=0.$$

We know that $2^n-2$ is congruent to $2^{n-j}-2$ modulo $p$. Thus, if we consider the sequence of factors of (*), we will have $s_{i,j}$, $(2-s_n)$, $\cdots$, $(2^j-2)-s_n$, $(2^{j+1}-2)-s_n$, $\cdots$, $(2^n-2)-s_n$ and if we reduce the sequence following $s_{i,j}$ modulo $p$, we see that it repeats itself after $j$ terms. Suppose $n=mj+i$, $1\leqq i\leqq j$. Then, when we reduce (*) modulo $p$ we will have

$$(-1)^m((2^j-2)-s_n)^m\cdots((2^{i+1}-2)-s_n)^m \cdot((2^i-2)-s_n)^{m+1}\cdots(2-s_n)^{m+1}(s_n)^m s_{i,j}=0$$

as an element of $k\Sigma_n$. This implies that $(w_ns_n)^{m+1}s_{i,j}=0$. By a remark above $w_n-1$ is a polynomial in $s_n$ which lacks a constant term. Thus $(w_n^2-w_n)^{m+1}=(w_n)^{m+1}(w_n-1)^{m+1}=(w_n)^{m+1}(s_n)^{m+1}H(s_n)=0$ in $k\Sigma_n$ where $H(x)$ is some polynomial in $k[x]$. Now let us set $e_n=\{w_n(p_{m+1}(w_n))\}^{m+1}$. From the foregoing, it is obvious that $e_n$ is an idempotent. We do not yet know it is nonzero and before we can show this, we must have the following theorem.

3.8. THEOREM. $d_ne_n=e_{n-1}d_n$.

PROOF. We assume $n=mj+i$, $1\leqq i\leqq j$. If $i>1$, we have

$$d_ne_n=d_n(w_n\{p_{m+1}(w_n)\})^{m+1}=\{w_{n-1}p_{m+1}(w_{n-1})\}^{m+1}d_n=e_{n-1}d_n.$$

If $i=1$, then $e_n=\{w_n(p_{m+1}(w_n))\}^{m+1}$ and $e_{n-1}=\{w_{n-1}p_m(w_{n-1})\}^m$. Now we note that $p_{m+1}(w_n)=p_m(w_n)+(1-w_n)^m$. Thus

$$\begin{aligned}d_ne_n&=d_n\{w_np_{m+1}(w_n)\}^{m+1}=\{w_{n-1}p_{m+1}(w_{n-1})\}^{m+1}d_n\\&=\Big\{\{w_{n-1}(p_m(w_{n-1}))\}^{m+1}+(m+1)(w_{n-1})^{m+1}(p_m(w_{n-1}))^m(1-w_{n-1})^m\\&\qquad+\cdots+(w_{n-1})^{m+1}(1-w_{n-1})^{m(m+1)}\Big\}d_n.\end{aligned}$$

Now every term of the form $a(w_{n-1})^{m+1}\{p_m(w_{n-1})\}^{m+1-t}(1-w_{n-1})^{tm}$ is zero since $1-w_{n-1}$ does not have a constant term and, thus, every term of the above form will have a factor of the form $(w_{n-1}s_{n-1})^m$ and this last is zero. Now the only possible nonzero term is the first. So we have

$$\begin{aligned}d_ne_n&=\{w_{n-1}(p_m(w_{n-1}))\}^{m+1}d_n\\&=(w_{n-1})^{m+1}\cdot p_m(w_{n-1})\cdot\{p_m(w_{n-1})\}^md_n\\&=(w_{n-1})^m\{p_m(w_{n-1})\}^md_n=e_{n-1}d_n\end{aligned}$$

since $(w_{n-1})^m=(w_{n-1})^{m+1}p_m(w_{n-1})$ by the remark after Proposition 3.6.

3.9. PROPOSITION. *$e_n$ is nonzero.*

PROOF. We shall proceed by induction. Since the field we are working over does not have characteristic two, it is easily seen that $e_2$ is not zero. Now let $n$ be the smallest integer with $e_n=0$. Then $e_{n-1}\neq 0$. Consider the commutative polynomial algebra over $k$ in a countable number of variables, say $k[x_1,\cdots]$. Then, since $e_n$ is zero, $e_n[x_1,\cdots,x_n]=0$. Thus

$$d_ne_n[x_1,\cdots,x_n]=e_{n-1}d_n[x_1,\cdots,x_n]=0.$$

Then

$$e_{n-1}(x_1[x_2, \cdots, x_n] - [x_1x_2, \cdots, x_n] + \cdots + (-1)^n x_n[x_1, \cdots, x_{n-1}]) = 0.$$

Since the terms inside the parentheses are linearly independent over $k\Sigma_n$, then we see $e_{n-1}[x_1, \cdots, x_ix_{i+1}, \cdots, x_n]=0$ for all $i$. Suppose $\pi$ and $\sigma$ are two elements of $\Sigma_n$ which appear in $e_{n-1}$. Then

$$\pi([x_1, \cdots, x_ix_{i+1}, \cdots, x_n]) = \sigma([x_1, \cdots, x_ix_{i+1}, \cdots, x_n])$$

if and only if $\pi=\sigma$. Thus, in order for $e_{n-1}([x_1, \cdots, x_ix_{i+1}, \cdots, x_n])$ to be zero, $e_{n-1}$ must be zero. This is a contradiction and we are done.

Using the $e_n$'s we have constructed, we see that there is a natural splitting of the complex $J_*A$ which is given in the $n$th dimension by $(J_*A)_n=e_n(J_*A)_n+(1-e_n)(J_*A)_n$. We would now like to determine the kernel of $e_n$. Apply the following filtration to $J_*A$. We let $F_iJ_*A$ be $J_*A$ if $i>0$, $F_0J_*A=J_*^2A$, and $F_iJ_*A$ be the subcomplex whose $n$th dimensional summand is $(s_n)^{-i}(J_*^2A)_n$ if $i<0$. Clearly each $F_iJ_*A$ is a complex and $F_iJ_*A$ contains $F_{i-1}J_*A$ and so is a filtration. We note that the complex $F_1J_*A/F_0J_*A$ is merely $Ch_*A$.

3.10. Proposition. *Let $n=mj+i$, $1\leqq i\leqq j$. Then $e_n(F_{-m}J_*A)_n=0$.*

Proof. Let $x\in(F_{-m}J_*A)_n$. Then $x=(s_n)^m(y)$ for $y$ some nontrivial shuffle. Then

$$e_n(x) = (w_n)^{m+1}(s_n)^m\{p_{m+1}(w_n)\}^{m+1}(y) = 0$$

since $(w_n)^{m+1}(s_n)^m s_{i,j}=0$ for all shuffles $s_{i,j}$.

3.11. Proposition. *Let $n=mj+i$, $1\leqq i\leqq j$. If $e_n(x)=0$, then $x\in(F_{-m}J_*A)_n$.*

Proof. We know that $e_n=1+\sum_{i=1}^t a_i(s_n)^i$ for some integer $t$. Therefore, if $e_n(x)=0$, $x=-\sum_{i=1}^t (s_n)^i(x)=s_n(x_1)$ for some $x_1$. By the same reasoning, $s_n(x_1)=(s_n)^2(x_2)$. Thus $x\in s_n\ (J_*^2A)_n$. Continuing in this manner, we find that $x\in(s_n)^m\ (J_*^2A)_n$ for every $m$ and the proposition is proved.

We can now state our main theorem.

3.12. Theorem. *Let $k$ be a ring containing a field of characteristic $p$ ($p\neq 2$). Let $j$ be the order of 2 in the group of units of $k$. Let $A$ be a commutative algebra over $k$ and $M$ a left $A$-module. Construct $J_*A$ and filter it as before. Let $n=mj+i$, $1\leqq i\leqq j$. Then there exist natural transformations*

$$\phi_n(A, M):\ \ \mathrm{Hoch}_n(A, M) \to H_n((J_*A/F_{-m}J_*A) \otimes_A M),$$
$$\phi^n(A, M):\ \ H^n(\mathrm{Hom}_A(J_*A/F_{-m}J_*A, M) \to \mathrm{Hoch}^n\ (A, M)$$

*such that* $\phi_n(A, M)$ *is a split epimorphism and* $\phi^n(A, M)$ *is a split monomorphism.*

The proof follows from the foregoing discussion.

It is also possible, using our filtration, to build a subcomplex of $J_*A$ called $K_*A$ and show that the homology of $K_*A$ is a natural direct summand of $J_*A$. We set $(K_*A)_n = (F_{-m}J_*A)_n$ if $n = mj+i$, $1 \leqq i \leqq j$. Then the proof that $K_*A$ is a complex is routine and the foregoing discussion obtains for us the following theorem.

3.13. Theorem. *Let k, A, j and M be as before. Then there exist natural transformations*

$$\psi_n(A, M):\ \ \mathrm{Hoch}_n(A, M) \to H_n((J_*A/K_*A) \otimes_A M),$$
$$\psi^n(A, M):\ \ H^n(\mathrm{Hom}_A(J_*A/K_*A, M)) \to \mathrm{Hoch}^n(A, M)$$

*such that* $\psi_n(A, M)$ *is a split epimorphism and* $\psi^n(A, M)$ *is a split monomorphism.*

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