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SOME EXTENSIONS OF THE MEHLER FORMULA[1]

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ABSTRACT. By using certain operational techniques, the authors prove an elegant unification of several extensions of the well-known Mehler formula for Hermite polynomials, given recently by L. Carlitz ([1], [2]). It is also shown how rapidly a number of Carlitz's formulas would follow from these considerations. The last section discusses a generalization involving the product of several Hermite polynomials of different arguments.

1. **Introduction.** In the theory of Hermite polynomials $\{H_n(z)\}$ defined by

$$\sum_{n=0}^{\infty} H_n(z) \frac{t^n}{n!} = \exp(2zt - t^2), \tag{1.1}$$

the bilinear generating function

$$\sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{t^n}{n!} = (1 - 4t^2)^{-1/2} \exp\left\{\frac{4xyt - 4(x^2 + y^2)t^2}{1 - 4t^2}\right\} \tag{1.2}$$

is well known as Mehler's formula [**3**, p. 198]. Recently, Carlitz ([**1**], [**2**]) has proved a number of extensions of (1.2). In particular, we recall here his elegant formula

$$\begin{aligned} \sum_{m,n,p=0}^{\infty} & H_{n+p}(x)H_{p+m}(y)H_{m+n}(z) \frac{u^m}{m!} \frac{v^n}{n!} \frac{w^p}{p!} \\ &= \Delta^{-1/2} \exp\left\{\sum x^2 - \frac{1}{\Delta}\left(\sum x^2 - 4 \sum u^2x^2 \right.\right. \\ &\qquad \left.\left. - 4 \sum wxy + 8 \sum uvxy\right)\right\}, \end{aligned} \tag{1.3}$$

where

$$\Delta = 1 - 4u^2 - 4v^2 - 4w^2 + 16uvw, \tag{1.4}$$

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and $\sum x^2$, $\sum u^2x^2$, $\sum wxy$, $\sum uvxy$ are symmetric functions in the indicated variables.

The two proofs of (1.3), given by Carlitz [**2**], seem to be long and involved. In the present note we first give a simple and direct proof of (1.3) by using certain operational techniques. We then proceed to derive the following result which generalizes (1.3) and a number of other extensions of (1.2) proved by Carlitz ([**1**, p. 45, equation (8)]; [**2**, pp. 117–118, equations (1.2), (1.4), (1.6), (1.7)]).

$$\begin{aligned}
&\sum_{m,n,p=0}^{\infty} H_{n+p+r}(x)H_{p+m+s}(y)H_{m+n}(z)\frac{u^m}{m!}\frac{v^n}{n!}\frac{w^p}{p!}\\
&\quad= \Delta^{-1/2}(1-4u^2)^{r/2}(1-4v^2)^{s/2}\\
&\quad\cdot \exp\left\{\sum x^2 - \frac{1}{\Delta}\left(\sum x^2 - 4\sum u^2x^2 - 4\sum wxy + 8\sum uvxy\right)\right\}\\
&\quad\cdot \sum_{k=0}^{\min(r,s)} 2^{2k}k!\binom{r}{k}\binom{s}{k}\left(\frac{w-2uv}{((1-4u^2)(1-4v^2))^{1/2}}\right)^k\\
&\quad\cdot H_{r-k}\left(\frac{(x-2vz)(1-4u^2)-2(y-2uz)(w-2uv)}{(\Delta(1-4u^2))^{1/2}}\right)\\
&\quad\cdot H_{s-k}\left(\frac{(y-2uz)(1-4v^2)-2(x-2vz)(w-2uv)}{(\Delta(1-4v^2))^{1/2}}\right),
\end{aligned} \tag{1.5}$$

where Δ is given by (1.4).

In our analysis we shall make use of a number of known results which we mention here for ready reference.

$$D_x^r H_n(x) = 2^r r!\binom{n}{r}H_{n-r}(x), \tag{1.6}$$

where, as usual, $D_z \equiv d/dz$.

$$\exp(tD_x)f(x) = f(x+t). \tag{1.7}$$

$$H_n(ax) = (-1/a)^n \exp(a^2x^2)D_x^n \exp(-a^2x^2), \tag{1.8}$$

which follows at once from Rodrigues' formula for the Hermite polynomial.

$$\exp(tD_x^2)\{\exp(-x^2)\} = (1+4t)^{-1/2}\exp\left\{\frac{-x^2}{1+4t}\right\}, \tag{1.9}$$

which is Glaisher's operational formula.

$$\begin{aligned}
&\exp(tD_xD_y)\{\exp(-a^2x^2-b^2y^2)\}\\
&\qquad= (1-4a^2b^2t^2)^{-1/2}\exp\left\{-a^2x^2 - \frac{(by-2a^2bxt)^2}{1-4a^2b^2t^2}\right\},
\end{aligned} \tag{1.10}$$

which was derived earlier by Singhal [**4**].

2. **Proof of (1.3).** Denoting the left-hand side of (1.3) by Ω, if we make use of (1.8) and (1.1), we get

$$\begin{aligned}
\Omega &= \sum_{m,n=0}^{\infty} H_{m+n}(z)\frac{u^m}{m!}\frac{v^n}{n!}\sum_{p=0}^{\infty} H_{n+p}(x)H_{p+m}(y)\frac{w^p}{p!} \\
&= \sum_{m,n=0}^{\infty} H_{m+n}(z)\frac{u^m}{m!}\frac{v^n}{n!} \\
&\quad \cdot \sum_{p=0}^{\infty} \exp(x^2+y^2)(-D_x)^{n+p}(-D_y)^{p+m}\exp(-x^2-y^2) \\
&= \exp(x^2+y^2) \\
&\quad \cdot \sum_{m,n=0}^{\infty} H_{m+n}(z)\frac{(-uD_y)^m}{m!}\frac{(-vD_x)^n}{n!}\exp(wD_xD_y)\exp(-x^2-y^2) \\
&= \exp(x^2+y^2) \\
&\quad \cdot \sum_{k=0}^{\infty} H_k(z)\frac{(-uD_y-vD_x)^k}{k!}\exp(wD_xD_y)\exp(-x^2-y^2) \\
&= \exp(x^2+y^2) \\
&\quad \cdot \exp\{-2uzD_y - 2vzD_x + (w-2uv)D_xD_y - u^2D_y^2 - v^2D_x^2\} \\
&\quad \cdot \{\exp(-x^2-y^2)\} \\
&= (1-4u^2)^{-1/2}(1-4v^2)^{-1/2}\exp(x^2+y^2) \\
&\quad \cdot \exp\{-2uzD_y - 2vzD_x + (w-2uv)D_xD_y\} \\
&\quad \cdot \exp\left\{-\frac{x^2}{1-4v^2} - \frac{y^2}{1-4u^2}\right\},
\end{aligned}$$

by virtue of Glaisher's formula (1.9).

Now we apply formulas (1.10) and (1.7), and we observe that

$$\begin{aligned}
\Omega &= \Delta^{-1/2}\exp(x^2+y^2)\exp\{-2uzD_y - 2vzD_x\} \\
&\quad \cdot \exp\left\{\frac{-x^2(1-4u^2) - y^2(1-4v^2) + 4xy(w-2uv)}{\Delta}\right\} \\
&= \Delta^{-1/2}\exp(x^2+y^2) \\
&\quad \cdot \exp\Big\{-\frac{1}{\Delta}((x-2vz)^2(1-4u^2) - (y-2uz)^2(1-4v^2) \\
&\qquad\qquad + 4(x-2vz)(y-2uz)(w-2uv))\Big\} \\
&= \Delta^{-1/2}\exp\left\{\sum x^2 - \frac{1}{\Delta}\left(\sum x^2 - 4\sum u^2x^2 - 4\sum wxy + 8\sum uvxy\right)\right\},
\end{aligned}$$

which completes the proof of Carlitz's formula (1.3).

3. **Proof of (1.5).** Following an analysis similar to the one used in the preceding section, it is readily seen that

$$\sum_{m,n,p=0}^{\infty} H_{n+p+r}(x)H_{p+m+s}(y)H_{m+n}(z)\frac{u^m}{m!}\frac{v^n}{n!}\frac{w^p}{p!}$$

$$= \Delta^{-1/2}\exp(x^2+y^2)\exp(-2uzD_y - 2vzD_x)$$

$$\cdot(-D_x)^r(-D_y)^s\exp\left\{-\frac{x^2}{1-4v^2}-\frac{(y(1-4v^2)-2x(w-uv))^2}{\Delta(1-4v^2)}\right\}$$

$$= \Delta^{-(s+1)/2}(1-4v^2)^{s/2}\exp(x^2+y^2)\exp\{-2uzD_y-2vzD_x\}$$

$$\cdot(-D_x)^r\exp\left\{-\frac{x^2}{1-4v^2}-\frac{(y(1-4v^2)-2x(w-2uv))^2}{\Delta(1-4v^2)}\right\}$$

$$\cdot H_s\left(\frac{y(1-4v^2)-2x(w-2uv)}{(\Delta(1-4v^2))^{1/2}}\right)$$

$$= \Delta^{-(s+1)/2}(1-4v^2)^{s/2}\exp(x^2+y^2)\exp\{-2uzD_y-2vzD_x\}$$

$$\cdot\sum_{k=0}^{r}(-1)^r\binom{r}{k}D_x^{r-k}$$

$$\cdot\exp\left\{-\frac{(x(1-4u^2)-2y(w-2uv))^2}{\Delta(1-4u^2)}-\frac{y^2}{1-4u^2}\right\}$$

$$\cdot D_x^kH_s\left(\frac{y(1-4v^2)-2x(w-2uv)}{(\Delta(1-4v^2))^{1/2}}\right)$$

$$= \Delta^{-(r+s+1)/2}(1-4u^2)^{r/2}(1-4v^2)^{s/2}\exp(x^2+y^2)$$

$$\cdot\exp(2uzD_y-2vzD_x)$$

$$\cdot\exp\left\{\frac{-x^2(1-4u^2)-y^2(1-4v^2)+4xy(w-2uv)}{\Delta}\right\}$$

$$\cdot\sum_{k=0}^{\min(r,s)}2^{2k}k!\binom{r}{k}\binom{s}{k}\left(\frac{w-2uv}{((1-4u^2)(1-4v^2))^{1/2}}\right)^k$$

$$\cdot H_{r-k}\left(\frac{x(1-4u^2)-2y(w-2uv)}{(\Delta(1-4u^2))^{1/2}}\right)$$

$$\cdot H_{s-k}\left(\frac{y(1-4v^2)-2x(w-2uv)}{(\Delta(1-4v^2))^{1/2}}\right),$$

which, in view of (1.7), leads us to the desired result (1.5).

4. **Particular cases of (1.5).** Evidently, (1.5) provides a generalization of Carlitz's formula (1.3) to which it would reduce when $r = s = 0$.

For $u = v = 0$, our formula (1.5) leads us to

$$\sum_{n=0}^{\infty} H_{n+r}(x)H_{n+s}(y)\frac{t^n}{n!}$$

$$= (1-4t^2)^{-(r+s+1)/2}\exp\left\{\frac{4xyt - 4(x^2+y^2)t^2}{1-4t^2}\right\} \tag{4.1}$$

$$\cdot \sum_{k=0}^{\min(r,s)} 2^{2k}k!\binom{r}{k}\binom{s}{k}t^k H_{r-k}\left(\frac{x-2yt}{(1-4t^2)^{1/2}}\right)H_{s-k}\left(\frac{y-2xt}{(1-4t^2)^{1/2}}\right),$$

which was proved by Carlitz [1] in a different way (see also [4]).

Yet another special case of (1.5) would seem to occur when u or v equals zero. Indeed we thus get the formula

$$\sum_{m,n=0}^{\infty} H_{m+n+r}(x)H_m(y)H_{n+s}(z)\frac{u^m}{m!}\frac{v^n}{n!}$$

$$= (1-4u^2-4v^2)^{-(r+s+1)/2}(1-4u^2)^{s/2}$$

$$\cdot \exp\left\{\frac{-4x^2(u^2+v^2)+4x(uy+vz)-4(uy+vz)^2}{1-4u^2-4v^2}\right\} \tag{4.2}$$

$$\cdot \sum_{k=0}^{\min(r,s)} 2^{2k}k!\binom{r}{k}\binom{s}{k}\left(\frac{v}{(1-4u^2)^{1/2}}\right)^k H_{r-k}\left(\frac{x-2uy-2vz}{(1-4u^2-4v^2)^{1/2}}\right)$$

$$\cdot H_{s-k}\left(\frac{z-4u^2z-2vx+4uvy}{((1-4u^2)(1-4u^2-4v^2))^{1/2}}\right),$$

which is believed to be new. For $s = 0$, formula (4.2) would further reduce to the elegant result

$$\sum_{m,n=0}^{\infty} H_{m+n+r}(x)H_m(y)H_n(z)\frac{u^m}{m!}\frac{v^n}{n!}$$

$$= (1-4u^2-4v^2)^{-(r+1)/2}$$

$$\cdot \exp\left\{\frac{-4x^2(u^2+v^2)+4x(uy+vz)-4(uy+vz)^2}{1-4u^2-4v^2}\right\} \tag{4.3}$$

$$\cdot H_r\left(\frac{x-2uy-2vz}{(1-4u^2-4v^2)^{1/2}}\right),$$

which provides a generalization of Carlitz's formula (see [2, p. 117, equation (1.2)])

$$\sum_{m,n=0}^{\infty} H_{m+n}(x)H_m(y)H_n(z)\frac{u^m}{m!}\frac{v^n}{n!} = (1-4u^2-4v^2)^{-1/2}$$

$$\cdot \exp\left\{\frac{-4x^2(u^2+v^2)+4x(uy+vz)-4(uy+vz)^2}{1-4u^2-4v^2}\right\} \tag{4.4}$$

which follows at once from (4.3) when $r = 0$. Lastly, on setting $w = 0$, (1.5) would give us

$$\sum_{m,n=0}^{\infty} H_{m+n}(x)H_{m+s}(y)H_{n+r}(z)\frac{u^m}{m!}\frac{v^n}{n!}$$

$$= (1-4u^2-4v^2)^{-(r+s+1)/2}(1-4u^2)^{r/2}(1-4v^2)^{s/2}$$

$$\cdot \exp\left\{\frac{-4x^2(u^2+v^2)+4x(uy+vz)-4(uy+vz)^2}{1-4u^2-4v^2}\right\}$$

(4.5)
$$\cdot \sum_{k=0}^{\min(r,s)} 2^{2k}k!\binom{r}{k}\binom{s}{k}\left(\frac{-2uv}{((1-4u^2)(1-4v^2))^{1/2}}\right)^k$$

$$\cdot H_{r-k}\left(\frac{z(1-4u^2)-2v(x-2uy)}{((1-4u^2)(1-4u^2-4v^2))^{1/2}}\right)$$

$$\cdot H_{s-k}\left(\frac{y(1-4v^2)-2u(x-2vz)}{((1-4v^2)(1-4u^2-4v^2))^{1/2}}\right).$$

Note that formula (4.5), with a different right-hand side, was also proved by Carlitz ([**2**, p. 118, equation (1.7)]).

It may be of interest to conclude with the remark that formulas (4.2) and (4.3) admit themselves of an elegant generalization in the form

$$\sum_{m,n_1,\cdots,n_k=0}^{\infty} H_{m+n_1+\cdots+n_k+r}(x)H_{m+s}(y)H_{n_1}(z_1)\cdots H_{n_k}(z_k)\frac{u^m}{m!}\frac{v_1^{n_1}}{n_1!}\cdots\frac{v_k^{n_k}}{n_k!}$$

$$= \left(1-4u^2-4\sum v_i^2\right)^{-(r+s+1)/2}\left(1-4\sum v_i^2\right)^{s/2}$$

$$\cdot \exp\left\{x^2-\frac{(x-2uy-2\sum v_iz_i)^2}{1-4u^2-4\sum v_i^2}\right\}$$

(4.6)
$$\cdot \sum_{k=0}^{\min(r,s)} 2^{2k}k!\binom{r}{k}\binom{s}{k}\left(\frac{u}{(1-4\sum v_i^2)^{1/2}}\right)^k$$

$$\cdot H_{r-k}\left(\frac{x-2uy-2\sum v_iz_i}{(1-4u^2-4\sum v_i^2)^{1/2}}\right)$$

$$\cdot H_{s-k}\left(\frac{y(1-4\sum v_i^2)-2u(x-2\sum v_iz_i)}{\{(1-4u^2-4\sum v_i^2)(1-4\sum v_i^2)\}^{1/2}}\right),$$

where the range of each i summation on the right-hand side is from $i = 1$ to $i = k$, $k = 1, 2, 3 \cdots$.

The proof of (4.6) would make use of the known formulas (1.6), (1.8), and (1.9) in a manner already illustrated in §§2 and 3.

Incidentally, (4.6) corresponds to Carlitz's formula (2.3) in [**2**, p. 120] when $r = s = u = 0$.

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