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SEMI-LOCAL-CONNECTEDNESS AND CUT POINTS IN METRIC CONTINUA

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ABSTRACT. In the first section of this paper, the notion of a space being rational at a point is generalized to what is here called quasi-rational at a point. It is shown that a compact metric continuum which is quasi-rational at each point of a dense subset of an open set is both connected im kleinen and semi-locally-connected on a dense subset of that open set. In the second section a G_δ set is constructed such that every point in the G_δ at which the space is not semi-locally-connected is a cut point. A condition is given for this G_δ set to be dense. This condition, in addition to requiring that the space be not semi-locally-connected at any point of a dense G_δ set gives a sufficient condition for the space to contain a G_δ set of cut points. The condition generalizes that given by Grace.

1. Throughout this paper M will be taken to be a compact metric continuum. Many of the lemmas, however, can be proven with less hypotheses. Lemma 2, for example, requires only that the sets P_x (defined below) be subcontinua of M. Compact Hausdorff is sufficient for this to happen [**4**].

Let x, y, and z be points of M (not necessarily distinct). The point x *cuts between* y *and* z in M when every subcontinuum of M which contains both y and z must also contain x. The point x is a *cut point* of M when x cuts between two points distinct from x. M is said to be *aposyndetic* (*semi-locally-connected*) *at* x *with respect to* y if and only if there is a subcontinuum of M with x (y) in its interior that does not contain y (x). M is *aposyndetic* (*semi-locally-connected*) *at* x when it is aposyndetic (semi-locally-connected) at x with respect to every other point. Finally, M is *connected im kleinen* at x when each neighborhood of x contains a closed neighborhood of x which is also connected. One should note that when M is connected im kleinen at a point, it is also aposyndetic at that point. For $x \in M$, P_x denotes $\{y \in M \mid M$ is not aposyndetic at y with respect to $x\}$, and for $T \subsetneq M$, P'_T denotes $\bigcap \{H \mid T \subseteq H^0$ and H is a

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subcontinuum of $M\}$. Clearly M is semi-locally-connected at x if and only if $P_x = \{x\}$. As noted above, P_x is a subcontinuum of M.

LEMMA 1.1. *If T is a subcontinuum of M and x is a point of M, then $x \in P'_T$ if and only if $P_x \cap T \neq \varnothing$.*

PROOF. Suppose $x \in P'_T$. If $P_x \cap T = \varnothing$, then M is aposyndetic at every point of T with respect to x. By the definition of aposyndetic and the compactness of T we see that T can be covered by the interior of a finite number of continua $A_1, \cdots, A_n$, where each A_i meets T and does not contain x. Now $T \cup \bigcup_{i=1}^n A_i$ is a subcontinuum of M containing T in its interior, and hence $x \in P'_T \subseteq T \cup \bigcup_{i=1}^n A_i$. Since $x \notin A_i$, we have $x \in T$, and so $x \in P_x \cap T = \varnothing$. This contradiction shows $P_x \cap T \neq \varnothing$.

Conversely, suppose that for some $x \in M, P_x \cap T \neq \varnothing$, say $y \in P_x \cap T$. Let $T \subseteq H^0$ where H is a subcontinuum of M. Then $y \in H^0$. Since $y \in P_x$, M is not aposyndetic at y with respect to x. It follows that $x \in H$. Thus $x \in P'_T$. □

LEMMA 1.2. *If T is a subcontinuum of M, then for $z \in (P'_T)^0$ and $y \in M - T$ we have $z \in P_y$ implies $y \in P_z$.*

PROOF. Suppose $y \notin P_z$ and $z \in P_y$. Then there is a continuum H containing y in its interior which does not contain z. Let U be an open neighborhood of y in $H^0 \cap (M - T)$, and let L be the component of $M - U$ containing T. Suppose $x \in (M - H) \cap P'_T$. Then in particular $P_x \cap T \neq \varnothing$. If $x \notin L$, then L is a proper subset of $P_x \cup L$. Hence $P_x \cap U \neq \varnothing$. Let $s \in P_x \cup U$, then $s \in H^0$ and thus $x \in H$. With this contradiction we conclude that $(M - H) \cap P'_T \subseteq L$. Thus $z \in (M - H) \cap (P'_T)^0 \subseteq L^0$ and, of course, $z \in P_y$. This implies $y \in L$ which contradicts the fact that $y \in U \subseteq M - L$. □

LEMMA 1.3. *Let V be an open point set of M. M is semi-locally-connected on a dense subset of V if and only if for each open point set W in V, there is a finite number of continua covering ∂W but not all of W.*

PROOF. That this condition is necessary is immediate, for if W is an open point set of V, then there is a point x in W at which M is semi-locally-connected. Thus M is aposyndetic at each point of ∂W with respect to x. Since ∂W is compact we can conclude there is a finite number of continua covering ∂W with their interior but not containing x.

Conversely let W be any open point set of V. We will find a point $x \in W$ at which M is semi-locally-connected. By the hypothesis we can choose open point sets W_i and continua $H_1^i, \cdots, H_{n_i}^i$ such that

(1) $W_1 \subseteq W$,

(2) $\partial W_i \subseteq \bigcup_{j=1}^{n_i} H_j^i$,

(3) $\bar{W}_{i+1} \subseteq W_i - \bigcup_{j=1}^{n_i} H_j^i$,

(4) $x, y \in W_i$ implies $d(x, y) \leqq 1/i$.

Let $x \in \bigcap W_i$. For $y \neq x$, choose k such that $y \notin N_{2/k}(x)$ ($N_r(x)$ is the open ball with center x and radius r). Then $x \in (W_k - \bigcup_{j=1}^{n_k} H_j^k) = U$, and $y \in M - \bar{U}$. Now each component of $M - \bar{U}$ meets ∂U which is in $\bigcup_{j=1}^{n_k} H_j^k$. Thus $M - U$ has only a finite number of components (each component of $M - U$ contains at least one H_j^k. Since $y \in M - \bar{U}$, y is in the interior of the component of $M - U$ containing y. Since this component does not contain x, M is semi-locally-connected at x with respect to y. It follows that M is semi-locally-connected at x which completes the proof. □

M is said to be *quasi-rational at* x if and only if for each open neighborhood W of x there is an open neighborhood U of x in W such that $W - U$ contains a closed set which is a countable union of continua and which separates U from $M - W$.

LEMMA 1.4. *If M is quasi-rational on a dense subset of an open point set V of M, then M is connected im kleinen on a dense subset of V.*

PROOF. Let W be an open point set in V. We will show W contains a point at which M is connected im kleinen. W contains a point at which M is quasi-rational. Thus there is an open point set U and continua $T_1, T_2, \cdots$ such that $\bigcup T_i \subseteq W - U$ is closed and separates U from $M - W$. Since each component of $M - \bigcup T_i$ meets some T_i, we see U is covered by a countable number of continua in W. One of these continua must contain an open subset of U. The above proof procedure allows us to verify that there are continua $H_1, H_2, \cdots$ in W such that for each positive integer i, $H_{i+1} \subseteq H_i^0$ and the diameter of H_i is $\leqq 1/i$. Let $x \in \bigcap H_i$. Then since $x \in H_i^0$ for each i and for each neighborhood G of x there exists an integer i such that H_i is contained in G, M is connected im kleinen at x. □

THEOREM 1.1. *If M is quasi-rational on a dense subset of an open point set V then M is semi-locally-connected on a dense subset of V.*

PROOF. Suppose not. By Lemma 1.3 there is an open point set W in V such that if ∂W is covered by a finite number of continua, then they cover all of W. This implies in particular that $P_x \cap \partial W \neq \varnothing$ for all $x \in W$. Now let U be an open point set in W which is separated from $M - W$ by a countable union of continua, $\bigcup T_i$, which is a closed subset of $W - U$. Since P_x is connected and $P_x \cap \partial W \neq \varnothing$ for $x \in W$, for $x \in U$ we have $P_x \cap \bigcup T_i \neq \varnothing$. Let $K_i = \{x \in U \mid P_x \cap T_i \neq \varnothing\}$.

It is easily seen that K_i is a closed subset of U (relative topology) for each i [**4**, Theorem 1]. Since the K_i's form a countable cover of U, it follows that for some i, $K_i^0 \neq \varnothing$. By Lemma 1.1 we see $K_i^0 \subseteq P'_{T_i} \cap (M - T_i)$. By Lemma 1.4 there is a point $x \in U \cap (P'_{T_i})^0$ at which M is aposyndetic. Since $P_x \cap \partial W \neq \varnothing$ and P_x is connected, there is a point $y \in U \cap (P'_{T_i})^0 \cap P_x$ different from x. By Lemma 1.2, $x \in P_y$ which contradicts the fact that M is aposyndetic at x with respect to y. □

2. In the following a G_δ set is constructed such that every point in the G_δ at which M is not semi-locally-connected is a cut point. Then it is proven that under certain conditions this G_δ is dense. In this section y is a fixed point of M. $C(x, i)$ is used to denote the component of $M - N_{1/i}(x)$ containing y, and when it is used it is assumed that $y \in M - N_{1/i}(x)$. Let $G_n = \{z \in M \mid \text{there is a point } x \in M \text{ and integers } i, j$ such that $d(x, z) < 1/n$, $i > n$, and $C(x, i) \subseteq C(z, j)^0\}$ and let $G = \bigcap G_n$.

LEMMA 2.1. *G_n is an open set for each n.*

PROOF. Let $z \in G_n$. There is a point x of M and integers i, j such that $d(x, z) < 1/n$, and $C(x, i) \subseteq C(z, j)^0$. $N_{1/n}(x) \cap N_{1/j}(z)$ is a neighborhood of z. For $s \in N_{1/n}(x) \cap N_{i/j}(z)$ we have $d(x, s) < 1/n$, and we can find a k so that $N_{1/k}(s) \subseteq N_{1/j}(z)$. Hence $C(z, j) \subseteq C(s, k)$. It follows that $C(s, i) \subseteq C(z, j)^0 \subseteq C(s, k)^0$, and thus $s \in G_n$. □

LEMMA 2.2. *If $z \in G$ and z is not a cut point, then M is semi-locally-connected at z.*

PROOF. Suppose $z \in G$ is not a cut point. For each positive integer n there exists a point x_n and integers i_n, j_n such that $d(x_n, z) < 1/n$, $i_n > n$ and $C(x_n, i_n) \subseteq C(z, j_n)^0$. Now $C(z, j_n)$ is a continuum not containing z, so $P_z \subseteq \bigcap_n (M - C(z, j_n))^0 \subseteq \bigcap_n (M - C(x_{n_j}, i_n))$. Suppose $s \in \bigcap_n (M - C(x_n, i_n))$ and $s \neq z$. Let H be a subcontinuum of M joining s to y and missing z. Choose k large enough so that $N_{1/k}(z) \cap H = \varnothing$. Then $H \subseteq C(z, k)$. Also choose p large enough so that $N_{1/i_p}(x_p) \subseteq N_{1/k}(z)$. Then $C(z, k) \subseteq C(x_p, i_p)$ and hence $s \in C(x_p, i_p)$. This contradiction shows $\bigcap (M - C(X_n, i_n)) \subseteq \{z\}$. Thus $P_x \subseteq \{z\}$ and M is semi-locally-connected at z. □

THEOREM 2.1. *Let V be an open set in M. Suppose for all continua T containing y we have that $(P'_T)^0 \cap (V - T) = \varnothing$, then $V \cap G$ is dense in V.*

PROOF. Suppose $W \subseteq V - G$ is an open point set of M. Let $x_1 \in W$. Choose $i_1 > 1$ such that $N_{1/i_1}(x_1) \subseteq W$ (with no loss of generality $y \notin W$). If there is an $x \in N_{1/i_1}(x_1)$ such that, for some j, $C(x_1, i_1) \subseteq C(x, j)^0$ then we let $x_2 = x$ and $i_2 = \max(j, 2, k)$ where k is such that

$$\overline{N_{1/k}(x)} \subseteq N_{1/i_1}(x_1).$$

Suppose for each positive integer n there exists a point x_n and an integer $i_n > n$ such that

$$C(x_n, i_n) \subseteq C(x_{n+1}, i_{n+1})^0 \quad \text{and} \quad \overline{N_{1/i_{n+1}}(x_{n+1})} \subseteq N_{1/i_n}(x_n).$$

Let $z \in \bigcap N_{1/i_n}(x_n)$. Then $x_1, x_2, x_3, \cdots$ converges to z. Since $z \in N_{1/i_{n+1}}(x_{n+1})$, for each positive integer n there is a j_n such that $N_{1/j_n}(z) \subseteq N_{1/i_{n+1}}(x_{n+1})$. Hence we conclude that $C(x_n, i_n) \subseteq C(x_{n+1}, i_{n+1})^0 \subseteq C(z, j_n)^0$. Since $d(x_n, z) < 1/n$ it follows that $z \in G_n$ for all n. But this says $z \in G \cap W$. We conclude that there must be an n such that $x \in N_{1/i_n}(x_n)$ implies $C(x_n, i_n) \nsubseteq C(x, j)^0$ for all j. Let $T = C(x_n, i_n)$. T is a subcontinuum of M containing y. Let $s \in N_{1/i_n}(x_n)$ and $T \subseteq H^0$ where H is a subcontinuum of M. If $s \notin H$, then there is a j such that $N_{1/j}(s) \subseteq M - H$. Hence $H \subseteq C(s, j)$. This says $C(x_n, i_n) = T \subseteq H^0 \subseteq C(s, j)^0$ which is a contradiction. Therefore $N_{1/i_n}(x_n) \subseteq H$. It follows that $N_{1/i_n}(x_n) \subseteq P'_T \cap (V - T)$, contradicting the fact that

$$(P_T)^0 \cap (V - T) = \varnothing. \quad \square$$

COROLLARY 2.1. *If M is not semi-locally-connected at any point of a dense G_δ subset of an open point set V, and if for any subcontinuum T of M containing y we have $(P'_T)^0 \cap (V - T) = \varnothing$. Then V contains a dense G_δ set of cut points.*

COROLLARY 2.2 (GRACE [2]). *Suppose V is an open set of M which contains a dense G_δ set G such that given any point x in G, M is locally peripherally aposyndetic at x and M is not semi-locally-connected at x. Then V contains a dense G_δ set of cut points.*

(M is locally peripherally aposyndetic at x when for $x \in U$, U open, there is an open set W such that $x \in W \subseteq U$ and M is aposyndetic at x with respect to each point of ∂W.)

PROOF. If V does not contain a dense G_δ set of cut points, then by Corollary 2.1 there is a continuum T such that $(P'_T)^0 \cap (V - T) \neq \varnothing$. Let $x \in (P'_T)^0 \cap (V - T)$ be a point at which M is both locally peripherally aposyndetic and not semi-locally-connected. Since M is not semi-locally-connected at x, there is an open set W such that $x \in W \subseteq (P'_T)^0 \cap (V - T)$ and $P_x \cap (M - W) \neq \varnothing$. Let U be open such that $x \in U \subseteq W$ and M is aposyndetic at x with respect to each point of ∂U. Since P_x is a continuum there is a $z \in P_x \cap \partial U$. By Lemma 1.2, $x \in P_z$. But this says M is not aposyndetic at x with respect to z and since $z \in \partial U$ we have a contradiction. $\square$

Jones [4, Theorem 15] has shown that a compact metric continuum M which is not semi-locally-connected at any of its points contains a dense set of cut points. Grace [1] posed the question whether a space M has a G_δ set of cut points. In particular, this would imply that the cardinality of the collection of cut points is c. Hagopian [3, Theorem 4] has shown that the latter must happen: If a compact metric continuum M is not semi-locally-connected at any point of a G_δ subset which is dense in M then the set of cut points in each open point set has cardinality c.

Suppose V is open in M and M is not semi-locally-connected at any point of a dense G_δ subset K of V. Let $V_1 = V \cap (\bigcup \{P_T'^0 - T \mid T$ is a subcontinuum of $M\})$ and let $V_2 = (V - V_1)^0$. By Corollary 2.1, V_2 contains a dense G_δ set of cut points. Although we cannot show that V_1 contains a dense G_δ set of cut points (which would answer Grace's question), we can strengthen Hagopian's result by proving that when $V_1 \neq \varnothing$, V_1 contains a nondegenerate continuum whose points are cut points. Assume $V_1 \neq \varnothing$.

THEOREM 2.2. *V_1 contains a dense G_δ set J such that for each $x \in J$ there is a nondegenerate subcontinuum H of M containing x such that each point of H cuts x from y.*

PROOF. As a special case of Grace's Theorem 2 [2] we have M contains a dense G_δ set I such that if $x \in I \cap P_z$ then z cuts x from y. Let $J = V_1 \cap I$. For $x \in J$ there is a subcontinuum T of M such that $x \in P_T'^0 - T$. Let H be any nondegenerate subcontinuum of P_x in $P_T'^0$ containing x. By Lemma 1.2, $z \in H$ implies $x \in P_z$. Since $x \in I$, z cuts x from y.

By choosing a nondegenerate subcontinuum K of H (the H of Theorem 2.2) which is contained in $V - \{x, y\}$, we have that each point of K is a cut point.

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