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QUASIHARMONIC CLASSIFICATION OF RIEMANNIAN MANIFOLDS[1]

MITSURU NAKAI AND LEO SARIO

ABSTRACT. In the study of the structure of the space of biharmonic functions it is often necessary to impose some nondegeneracy condition on the base manifold with respect to quasiharmonic functions (cf. [2], [4]). For this reason it is useful to introduce various quasiharmonically degenerate classes of Riemannian manifolds and to investigate relations among them. This is the purpose of the present note.

1. **Quasiharmonic degeneracy.** Let R be a noncompact orientable C^∞-manifold of dimension $m \geqq 2$ with C^∞ Riemannian metric $ds^2 = g_{ij}\,dx^i dx^j$. The corresponding Laplace-Beltrami operator is

$$(1) \qquad \Delta\cdot = -g^{-1/2}\sum_{i=1}^{m}\frac{\partial}{\partial x^i}\left(\sum_{j=1}^{m} g^{1/2} g^{ij}\frac{\partial\cdot}{\partial x^j}\right),$$

where $g = \det(g_{ij})$ and $(g^{ij}) = (g_{ij})^{-1}$. We call a function u *quasiharmonic* if $\Delta u = \text{const} \neq 0$, and denote by $Q = Q(R)$ the class of quasiharmonic functions u on R normalized by $\Delta u = 1$. Such functions are superharmonic on R.

Following [5], we denote the classes of nonnegative, bounded, and Dirichlet finite functions by P, B, and D respectively, and we set $BD = B \cap D$. Similarly we write QX ($X = P, B, D$, or BD) for $Q \cap X$. We are interested in the question as to when $QX = \varnothing$.

2. **Characterization of null classes.** We denote by O_{QX} the class of Riemannian manifolds R on which $QX(R) = \varnothing$, and by O_G the class of parabolic manifolds.

Let $G(x, y) = G_R(x, y)$ be the harmonic Green's function on $R \notin O_G$, and set $G(x, y) = \infty$ on $R \in O_G$. Denote by dy the Riemannian volume element $(g(y))^{1/2}\,dy^1 \cdots dy^m$.

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THEOREM 1. *The classes O_{QX} are characterized in terms of $G(x, y)$ as follows*:

$$
(2)\qquad
\begin{aligned}
&R \in O_{QP} \quad \textit{if and only if} \quad \int_R G(x, y)\, dy = \infty;\\
&R \in O_{QB} \quad \textit{if and only if} \quad a = \sup_{x\in R} \int_R G(x, y)\, dy = \infty;\\
&R \in O_{QD} \quad \textit{if and only if} \quad b = \int_{R\times R} G(x, y)\, dx\, dy = \infty;\\
&R \in O_{QBD} \quad \textit{if and only if} \quad a = \infty \quad \textit{or} \quad b = \infty.
\end{aligned}
$$

3. **Proof.** Let $u \in Q(R)$ and take a regular subregion Ω of R containing a given point x. Denote by H_u^Ω the harmonic solution of the Dirichlet problem on Ω with boundary values u. By Stokes' formula,

$$
(3)\qquad
\begin{aligned}
&\int_{\Omega-\bar{B}} [(u(y) - H_u^\Omega(y))\, \Delta_y G_\Omega(x, y) - G_\Omega(x, y)\, \Delta_y(u(y) - H_u^\Omega(y))]\, dy\\
&\qquad = -\int_{\partial\Omega-\partial B} [(u(y) - H_u^\Omega(y)) * d_y G_\Omega(x, y)\\
&\qquad\qquad - G_\Omega(x, y) * d_y(u(y) - H_u^\Omega(y))],
\end{aligned}
$$

with B a small geodesic ball about x of radius ε. On letting $\varepsilon \to 0$, we obtain the Riesz representation

$$
(4)\qquad u(x) = H_u^\Omega(x) + \int_\Omega G_\Omega(x, y)\, dy
$$

on Ω (cf., e.g., [1]). Again by Stokes' formula,

$$
(5)\qquad
\begin{aligned}
D_\Omega\left(\int_\Omega G_\Omega(\cdot, y)\, dy\right) &= \int_\Omega \left(\int_\Omega G_\Omega(x, y)\, dy \cdot \Delta_x \int_\Omega G_\Omega(x, y)\, dy\right) dx\\
&= \int_{\Omega\times\Omega} G_\Omega(x, y)\, dx\, dy.
\end{aligned}
$$

We conclude that

$$
(6)\qquad D_\Omega(u) = D_\Omega(H_u^\Omega) + \int_{\Omega\times\Omega} G_\Omega(x, y)\, dx\, dy.
$$

If $\int_R G(x, y)\, dy$ exists, then it is of class C^2, and

$$
\Delta_x \int_R G(x, y)\, dy = 1
$$

(cf., e.g., [3]). On letting $\Omega \to R$ in (5), we obtain

$$
(7)\qquad D_R\left(\int_R G(\cdot, y)\, dy\right) = \int_{R\times R} G(x, y)\, dx\, dy \leqq \infty.
$$

Suppose there exists a $u \in QP$. Since u is positive and superharmonic on R, $h(x) = \lim_{\Omega \to R} H_u^\Omega(x)$ exists and a fortiori

$$\int_R G(x, y)\, dy = \lim_{\Omega \to R} \int_\Omega G_\Omega(x, y)\, dy = u(x) - h(x) < \infty.$$

If $u \in QB$, then since $|H_u^\Omega(x)| \leqq \sup_{\partial\Omega} |u|$ for every $x \in \Omega$, (4) implies

$$\int_\Omega G_\Omega(\cdot, y)\, dy \leqq 2 \sup_R |u|$$

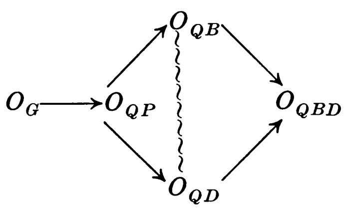
and consequently $a < \infty$. If $u \in QD$, then (6) and (7) give $b < \infty$. A fortiori, if $u \in QBD$, then $a < \infty$ and $b < \infty$.

Conversely let $v(x) = \int_R G(x, y)\, dy$. If it is finite or bounded on R, then $v \in QP$ or QB. If $b < \infty$, then $v(x)$ is finite and (7) implies $v \in QD$. Consequently, if $a < \infty$ and $b < \infty$, then $v \in QBD$.

4. **Strict inclusion relations.** By means of the characterizations in Theorem 1 we shall prove:

THEOREM 2. *The following strict inclusion relations are valid:*

(8)



Here $X \to Y$ stands for $X \subsetneqq Y$, and $X \sim\sim\sim Y$ means $X \not\subset Y$ and $Y \not\subset X$.

5. **An auxiliary metric.** The inclusion $O_G \subset O_{QP} \subset O_{QB}$, $O_{QP} \subset O_{QD}$, and $O_{QB} \cup O_{QD} \subset O_{QBD}$ follow immediately from (2). The simplest example showing the strictness of the inclusion $O_G \subsetneqq O_{QP}$ is the Euclidean space E^m $(m \geqq 3)$. To construct examples showing the strictness of the other inclusion relations, we make the following preliminary observation.

Take a hyperbolic region S in the z-plane. Denote by Δ_e the Laplacian $-4\partial^2/\partial z \partial \bar{z}$ and by $G(z, \zeta)$ the Green's function on S. Let v be a C^∞ superharmonic function on S with $\Delta_e v > 0$ and suppose that v is a potential, i.e., the greatest harmonic minorant of v is zero. Let $\lambda(z) = \Delta_e v(z)$. Consider the Riemannian manifold R with base manifold S and metric $(\lambda(z))^{1/2}\,|dz|$. The Laplace-Beltrami operator Δ is simply $\Delta = \lambda^{-1}\Delta_e$. Thus a function is harmonic on R if and only if it is so on S, i.e., $H(R) = H(S)$. The volume element dV on R is given by $dV(z) = \frac{1}{2} i\lambda(z)\, dz \wedge d\bar{z}$.

In view of $v \in C^\infty$ we obtain on replacing u by v in (3) and letting $\varepsilon \to 0$,

$$v(z) = H_v^\Omega(z) + \frac{i}{2}\int_\Omega G_\Omega(z,\zeta)\,\Delta_e v(\zeta)\,d\zeta \wedge d\bar{\zeta}.$$

Since v is a potential, we have $\lim_{\Omega\to R} H_v^\Omega(z) = 0$. Therefore

$$\int_R G(z,\zeta)\,dV(\zeta) = v(z) \tag{9}$$

and

$$\int_{R\times R} G(z,\zeta)\,dV(z)\,dV(\zeta) = \frac{i}{2}\int_R v(z)\,\Delta_e v(z)\,dz \wedge d\bar{z}. \tag{10}$$

6. **Example 1.** Let R be as in §5, and choose

$$S = \{0 < |z| < 1\}, \qquad v(z) = \log(1 - \log|z|). \tag{11}$$

Then

$$\Delta_e v(z) = |z|^{-2}(1 - \log|z|)^{-2} > 0 \tag{12}$$

and, for $\Omega(\rho) = \{\rho < |z| < 1\}$,

$$H_v^{\Omega(\rho)}(z) = \frac{\log(1-\log\rho)\cdot\log|z|}{\log\rho}.$$

Since this tends to zero as $\rho \to 0$, R qualifies as a manifold of §5. By (9) we have

$$\int_R G(z,\zeta)\,dV(\zeta) = \log(1 - \log|z|),$$

which is finite but not bounded on R. This with (2) implies that $R \notin O_{QP}$ and $R \in O_{QB} \subset O_{QBD}$.

Similarly (10) gives

$$\int_{R\times R} G(z,\zeta)\,dV(z)\,dV(\zeta)$$
$$= \frac{i}{2}\int_{R\times R} \log(1-\log|z|)\cdot|z|^{-2}(\log|z|)^{-2}\,dz\wedge d\bar{z} < \infty.$$

Again by (2) we conclude that $R \notin O_{QD}$. The example (11) thus serves to show that $O_{QP} \subsetneqq O_{QB}$, $O_{QD} \subsetneqq O_{QBD}$, and $O_{QB} \not\subset O_{QD}$.

7. **Example 2.** Let R be again as in §5 but now choose

$$S = \{0 < |z| < 1\}, \qquad v(z) = (1 - |z|)^{1/2}. \tag{13}$$

Then

$$\Delta_e v(z) = 4^{-1}(2 - |z|)\,|z|^{-1}\,(1 - |z|)^{-1/2-1} > 0. \tag{14}$$

Since v vanishes on $|z| = 1$ and is bounded on R, it is clearly a potential on R. Therefore R satisfies the condition of §5. By (9), we have

$$\int_R G(z, \zeta)\, dV(\zeta) = (1 - |z|)^{1/2}.$$

In view of (2) we conclude that $R \notin O_{QB}$ and a fortiori $R \notin O_{QP}$. Moreover it follows from (10) that

$$\int_{R\times R} G(z,\zeta)\, dV(z)\, dV(\zeta) = \frac{i}{2}\int_{R\times R} (1 - |z|)^{1/2} \cdot \Delta_e(1 - |z|)^{1/2}\, dz \wedge d\bar{z} = \infty.$$

This with (2) gives $R \in O_{QD} \subset O_{QBD}$. The example (14) thus yields the relations $O_{QP} \subsetneqq O_{QD}$, $O_{QB} \subsetneqq O_{QBD}$, and $O_{QD} \not\subset O_{QB}$.

The proof of Theorem 2 is herewith complete.

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Department of Mathematics, University of California, Los Angeles, California 90024

Current address: 52 Eguchi, Hinaga, Chita, Aichi 478, Japan

Current address: 521 Georgina Avenue, Santa Monica, California 90402