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THE COUNTING VECTOR OF A SIMPLE GAME[1]

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ABSTRACT. The counting vector of a simple game is the vector $f=(f(1),f(2),\cdots,f(n))$ where $f(i)$ is the number of winning coalitions containing the player i. In this paper, we show that the counting vector of a weighted majority game determines the game uniquely. With the aid of the counting vector we find an upper bound on the number of weighted majority games.

1. **Preliminaries on simple games.** A simple game is a pair $G=(N;W)$, where $N=\{1,2,\cdots,n\}$ is a set of n members and W is a set of subsets of N. The members of N are called players; subsets of N are called coalitions. The elements of W are called winning coalitions. A simple game is called monotone if every superset of a winning coalition is itself a winning coalition. A weighted majority game is a simple game for which there exist n nonnegative numbers $w_1, w_2, \cdots, w_n$ and a positive number q, such that, S is a winning coalition if and only if $w(S)=\sum_{i\in S} w_i \geqq q$. $w=[w_1, w_2, \cdots, w_n; q]$ is called the representation of the game. A weighted majority game is denoted by $G=(N;w)$ where w is its representation.

G is called constant-sum if for each coalition S exactly one of the two coalitions S and $N-S$ is winning.

2. **The counting vector theorem.** Given a simple game $G=(N;W)$, we denote by $f(i)$ the number of winning coalitions containing player i, and by f the vector $f=(f(1),f(2),\cdots,f(n))$ called the counting vector of the game.

THEOREM 2.1. *Let*

$$G_1=(N;[w^{(1)}, w_2^{(1)}, \cdots, w_n^{(1)}; q^{(1)}])$$

and

$$G_2=(N;[w_1^{(2)}, w_2^{(2)}, \cdots, w_n^{(2)}; q^{(2)}])$$

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be two weighted majority games. Their counting vectors f_1 *and* f_2 *are equal if and only if* $G_1 = G_2$.

PROOF. Suppose that $G_1 \neq G_2$ but $f_1 = f_2$, i.e., for every $i \in N$,

$$f_1(i) = f_2(i). \tag{2.1}$$

(We may assume that $f \neq 0$ since otherwise we have the trivial game $(N; \phi)$.) Let W_1, W_2 be the sets of winning coalitions in G_1, G_2 respectively. We set $\Omega_1 = W_1 - W_2$, $\Omega_2 = W_2 - W_1$. If $W_k \subset W_p$ $(k;\ p = 1, 2,\ k \neq p)$, then $f_k(i) \leqq f_p(i)$ for every i and inequality holds for at least one player i_0, in contradiction to (2.1); hence,

$$\Omega_1 \neq \phi, \qquad \Omega_2 \neq \phi. \tag{2.2}$$

For each $S \in \Omega_1$ and $T \in \Omega_2$ we have

$$w^{(1)}(S) > w^{(1)}(T). \tag{2.3}$$

Summing these inequalities over Ω_2 for a fixed $S \in \Omega_1$, we have

$$|\Omega_2|\, w^{(1)}(S) > \sum_{T \in \Omega_2} w^{(1)}(T) \tag{2.4}$$

where $|\Omega|$ is the number of elements of Ω. Summing all the inequalities (2.4) over Ω_1, we have

$$|\Omega_2| \sum_{S \in \Omega_1} w^{(1)}(S) > |\Omega_1| \sum_{T \in \Omega_2} w^{(1)}(T). \tag{2.5}$$

Let $\phi_k(i)$ be the number of coalitions in Ω_k containing player i. Since $\Omega_1 = W_1 - (W_1 \cap W_2)$ and $\Omega_2 = W_2 - (W_1 \cap W_2)$, $f_1(i) = f_2(i)$ implies $\phi_1(i) = \phi_2(i)$. Denoting the common value by $\phi(i)$, we have

$$\sum_{S \in \Omega_1} w^{(1)}(S) = \sum_{i \in N} \phi(i) w_i^{(1)}, \tag{2.6}$$

$$\sum_{T \in \Omega_2} w^{(1)}(T) = \sum_{i \in N} \phi(i) w_i^{(1)}. \tag{2.7}$$

Since $q^{(1)}$ is positive and $\Omega_1 \neq \phi$, $\Omega_2 \neq \phi$, we have, from (2.5), (2.6), and (2.7),

$$|\Omega_1| < |\Omega_2|. \tag{2.8}$$

Using the representation $[w_1^{(2)}, w_2^{(2)}, \cdots, w_n^{(2)}; q^{(2)}]$ of G_2 we have

$$|\Omega_2| < |\Omega_1| \tag{2.9}$$

in contradiction to (2.8) and hence $f_1 \neq f_2$.

COROLLARY 2.1. *Let $G_1 = (N; W_1)$ be a weighted majority game and $G_2 = (N; W_2)$ a simple game. If $G_1 \neq G_2$ and*

$$|W_1| \geqq |W_2|, \tag{2.10}$$

then $f_1 \neq f_2$.

PROOF. (2.10) implies

$$|\Omega_1| \geqq |\Omega_2|. \tag{2.11}$$

If $f_1 = f_2$ then as in the proof of Theorem 2.1, $|\Omega_1| < |\Omega_2|$. Hence $f_1 \neq f_2$.

COROLLARY 2.2. *Let G_1 be a constant-sum weighted majority game and G_2 a constant-sum simple game. If $G_1 \neq G_2$, then $f_1 \neq f_2$.*

PROOF. In any constant-sum simple game, the number of winning coalitions is 2^{n-1} (n being the number of players), hence $|W_1| = |W_2|$. The assumptions of Corollary 2.1 are valid, hence $f_1 \neq f_2$.

3. **An upper bound on the number of weighted majority games.** In **[1]**, J. R. Isbell raised the problem of finding an upper bound on the number of weighted majority games. No answer was given, except for the bound 2^{2^n}, which is the number of sets of coalitions in N. Using the counting vector concept a smaller value is obtainable for the upper bound.

It is evident that for every i, $0 \leqq f(i) \leqq 2^{n-1}$. If $f(i) = 0$ for some i, then $N \notin W$; hence $W = \phi$, i.e., the game is the trivial game $G_0 = (N; \phi)$. For all other games $1 \leqq f(i) \leqq 2^{n-1}$. The number of vectors

$$a = (a_1, a_2, \cdots, a_n), \qquad a_i = 1, 2, \cdots, 2^{n-1}, \quad i = 1, 2, \cdots, n, \tag{3.1}$$

is $2^{n(n-1)}$. Since not every vector of the type (3.1) is a counting vector, there are less than $2^{n(n-1)}$ weighted majority n-person games.

Two games, $G_1 = (N; W_1)$ and $G_2 = (N; W_2)$, are said to be equivalent if there exists a permutation π of N such that

$$W_2 = \{\pi S \colon S \in W_1\} \tag{3.2}$$

where $\pi S = \{\pi(i) \colon i \in S\}$.

It is evident that this relation is an equivalence. If we identify equivalent games, f is no longer a vector but an unordered set of n numbers. Hence their number is less than $\binom{2^{n-1}+n-1}{n}$, the number of n-selections of 2^{n-1} given elements.

The following example shows that there exist two nonequivalent monotone games which have the same counting vector.

EXAMPLE. Let N be a set of seven players $\{1, 2, 3, 4, 5, 6, 7\}$. W_1 consists of all 4, 5, 6, and 7 player coalitions and of the following 3 players coalitions: (2, 5, 3), (3, 7, 1), (1, 4, 2), (2, 6, 7), (3, 6, 4), (1, 6, 5), and

(4, 5, 7). W_2 consists of all 4, 5, 6, and 7 player coalitions and of the following 3 player ones: (2, 4, 3), (3, 7, 1), (1, 5, 2), (2, 6, 7), (3, 6, 4), (1, 6, 5), and (4, 5, 7).

The two games $(N; W_1)$ and $(N; W_2)$ are monotone and it is easy to see that they have the same counting vector. However, W_1 contains no two disjoint 3 player coalitions while W_2 does, and thus they are not equivalent.

4. **The desirability relation.** Two players i and j, in a simple game $(N; W)$ are called symmetric if for every coalition S not containing i nor j, $S \cup \{i\}$ is winning if and only if $S \cup \{j\}$ is winning; this is denoted by $i \sim j$. If for every coalition S not containing i nor j, $S \cup \{j\} \in W$ implies $S \cup \{i\} \in W$, then i is said to be more desirable than j; this is denoted by $i \gtrsim j$:

If $i \gtrsim j$ and $j \gtrsim i$ then $i \sim j$. If $i \gtrsim j$ but i and j are not symmetric, we say that i is strictly more desirable than j; this is denoted by $i \succ j$. In a weighted majority game with a representation $[w_1, w_2, \cdots, w_n; q]$, $w_1 \geqq w_j$ implies $i \gtrsim j$ and there exists a representation $[w_1^*, w_2^*, \cdots, w_n^*; q^*]$ such that $i \sim j$ implies $w_i^* = w^*$[3]. The counting vector strictly preserves the desirability relation, i.e., $f(i) > f(j)$ implies $i \succ j$ and $f(i) = f(j)$ implies $i \sim j$ [**2**]. This enables us to reduce the number of unknowns when seeking a representation of a given weighted majority game.

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