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A NONARCHIMEDEAN THEORY OF ANALYTIC CONTINUATION IN SEVERAL VARIABLES

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ABSTRACT. Recently B. Dwork proved the validity of the functional equation, conjectured by A. Weil, for a nonsingular projective hypersurface defined over a finite field. The proof made use of work of M. Krasner, wherein a uniqueness theorem for an analog of analytic continuation in ultrametric spaces is proved. The methods involved give information concerning the behavior of the undetermined factor ± 1 in the functional equation for such a hypersurface if one of the coefficients of the polynomial is varied. In this paper, Krasner's result is extended to a uniqueness theorem for analytic elements in n variables. This result will be applied to the Weil zeta function in a later work.

1. **Preliminaries.** Let $\mathfrak{K}$ be an algebraically closed field complete with respect to a nonarchimedean rank one valuation $x \to \operatorname{ord} x$ with value group $\mathfrak{G} \subset \boldsymbol{R}$, where $\boldsymbol{R}$ denotes the additive group of real numbers. We shall assume that $\mathfrak{G}$ is dense in $\boldsymbol{R}$. For $b \in \boldsymbol{R}$, we define $\Gamma_b = \{\xi \in \mathfrak{K}: \operatorname{ord} \xi = b\}$. Let $\mathfrak{O}$ denote the valuation ring of $\mathfrak{K}$, $\mathfrak{O} = \bigcup_{b \geq 0} \Gamma_b$, and let $\mathfrak{P}$ denote the ideals of nonunits in $\mathfrak{O}$, $\mathfrak{P} = \bigcup_{b>0} \Gamma_b$. It will occasionally be convenient to use the notation $|x| = p^{-\operatorname{ord} x}$, where p is the characteristic of the residue class field of $\mathfrak{K}$, denoted by k.

The following definition is due to Krasner **[2]**.

DEFINITION 1.1. Let D be a subset of the "projective field" $\mathfrak{K}^* = \mathfrak{K} \cup \{\infty^*\}$. We say that D is a *quasi-connected domain* of $\mathfrak{K}^*$ if, for every $\alpha \in D \cap \mathfrak{K}$, the following property is satisfied: for every $\xi \in D$, the set of real numbers

$$H_\xi = \{|x - \alpha| : x \in \mathfrak{K} - D, |x - \alpha| < |\xi - \alpha|\}$$

is a finite set.

LEMMA 1.2. *Let* $\zeta_1, \zeta_2, \cdots, \zeta_r$ *be distinct elements of* $\mathfrak{O}$; *then there is an element* ξ *of* $\mathfrak{O}$ *such that* $|\xi - \zeta_i| = 1$ *for* $i = 1, 2, \cdots, r$.

This is a special case of Lemma 1 of **[3]**, and so we may omit the proof.

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PROPOSITION 1.3. *Let $f(x) \in \mathfrak{O}[x]$, $f \not\equiv 0$. For any positive number δ, the sets*

$$W_\delta(f) = \{\xi \in \mathfrak{O} : |f(\xi)| > \delta\}, \qquad W_\delta^\#(f) = \{\xi \in \mathfrak{O} : |f(\xi)| \geqq \delta\}$$

are quasi-connected.

PROOF. Let $f(x) = (x - \zeta_1)^{e_1} \cdots (x - \zeta_r)^{e_r}(x\beta_1 - 1)^{c_1} \cdots (x\beta_s - 1)^{c_s}$, $\zeta_1, \zeta_2, \cdots, \zeta_r$ distinct elements of $\mathfrak{O}$, $\beta_1, \beta_2, \cdots, \beta_s$ distinct nonunits in $\mathfrak{O}$. For $\delta \geqq 0$, let $\mathfrak{R}_\delta$, $\mathfrak{R}_\delta^\#$ be sets of real r-tuples defined by

$$\mathfrak{R}_\delta = \{(\delta_1, \cdots, \delta_r) : \delta_1^{e_1} \cdots \delta_r^{e_r} > \delta, 0 \leqq \delta_i \leqq 1, i = 1, \cdots, r\},$$
$$\mathfrak{R}_\delta^\# = \{(\delta_1, \cdots, \delta_r) : \delta_1^{e_1} \cdots \delta_r^{e_r} \geqq \delta, 0 \leqq \delta_i \leqq 1, i = 1, \cdots, r\},$$

and, for any r-tuple $(\delta_1, \cdots, \delta_r)$, let $W(\delta_1, \cdots, \delta_r) = \{\xi \in \mathfrak{O} : |\xi - \zeta_i| \geqq \delta_i, i = 1, 2, \cdots, r\}$. Since, as is clear from Definition 1.1, a disk from which finitely many (open or closed) disks have been removed is a quasi-connected domain, it follows that, for any r-tuple $(\delta_1, \cdots, \delta_r)$, the set $W(\delta_1, \cdots, \delta_r)$ is quasi-connected.

Let us consider the collections

$$\mathfrak{C}_\delta = \{W(\delta_1, \cdots, \delta_r) : (\delta_1, \cdots, \delta_r) \in \mathfrak{R}_\delta\},$$
$$\mathfrak{C}_\delta^\# = \{W(\delta_1, \cdots, \delta_r) : (\delta_1, \cdots, \delta_r) \in \mathfrak{R}_\delta^\#\}.$$

It is noted that, for any δ, $\mathfrak{C}_\delta$ is a subfamily of $\mathfrak{C}_\delta^\#$, and that $\mathfrak{C}_\delta$ (respectively $\mathfrak{C}_\delta^\#$) is an empty family of sets if $\delta \geqq 1$ (respectively $\delta > 1$). We now recall that, in the terminology of Krasner, a family F of sets is said to be *linked* if any two sets A, B of F can be joined by a *chain*, that is to say a finite collection $A = C_0, C_1, \cdots, C_m = B$ of sets of the family such that any two consecutive terms C_{i-1}, C_i are nondisjoint, and we assert that the collections $\mathfrak{C}_\delta$, $\mathfrak{C}_\delta^\#$ are either empty or linked families of quasi-connected sets. In fact, we are able to prove a stronger statement, namely that for any choice of δ in the closed unit interval, there is an element $\xi \in \mathfrak{O}$ common to each member of the family $\mathfrak{C}_\delta^\#$. For, according to Lemma 1.2, an element ξ of $\mathfrak{O}$ may be chosen satisfying $|\xi - \zeta_i| = 1, i = 1, 2, \cdots, r$, and therefore, since $(\delta_1, \cdots, \delta_r) \in \mathfrak{C}_\delta^\#$ entails $\delta_i \leqq 1$ for all i, the assertion follows. But then, by a theorem of Krasner in the cited reference, the sets $\bigcup_{W \in \mathfrak{C}_\delta} W$, $\bigcup_{W \in \mathfrak{C}_\delta^\#} W$ are quasi-connected, for any nonnegative δ (note that the empty set is trivially a quasi-connected domain). Our desired result then follows from the observations that these latter unions are the sets $W_\delta(f)$ and $W_\delta^\#(f)$, respectively.

DEFINITION 1.4. Let V be a subset of $\mathfrak{K}^n$, j a positive integer, $1 \leqq j \leqq n$, and $(a_1, a_2, \cdots, a_{n-1}) \in \mathfrak{K}^{n-1}$. The symbol $V^{(j)}(a_1, \cdots, a_{n-1})$ denotes the subset of $\mathfrak{K}$ defined by

$$V^{(j)}(a_1, \cdots, a_{n-1}) = \{\alpha \in \mathfrak{K} : (a_1, \cdots, a_{j-1}, \alpha, a_j, \cdots, a_{n-1}) \in V\}.$$

If V has the property that, for each integer j, $1 \leqq j \leqq n$, and for each $(n-1)$-tuple $(a_1, \cdots, a_{n-1}) \in \mathfrak{K}^{n-1}$, the set $V^{(j)}(a_1, \cdots, a_{n-1})$ is a quasi-connected domain, the set V is said to be *axially quasi-connected.*

COROLLARY 1.5. *For* $R(X_1, X_2, \cdots, X_n) \in \mathfrak{O}[X_1, X_2, \cdots, X_n]$ *let* $W = \{(\xi_1, \xi_2, \cdots, \xi_n) \in \mathfrak{O}^n : \text{ord } R(\xi_1, \xi_2, \cdots, \xi_n) = 0\}$. *Then the set* W *is axially quasi-connected.*

PROOF. Let j be any integer between 1 and n, and let $R_j^*(x) \in \mathfrak{O}[x]$ be defined by

$$R_j^*(x) = R(a_1, \cdots, a_{j-1}, x, a_j, \cdots, a_{n-1}),$$

where $a_1, \cdots, a_{n-1}$ are arbitrarily chosen elements of $\mathfrak{O}$; then $W^{(j)}(a_1, \cdots, a_{n-1})$ is either empty or equal to $W_1^{\#}(R_j^*)$, and the preceding proposition applies.

2. **Uniqueness theorem.** In this section, a uniqueness theorem for analytic elements in several variables, generalizing the one-variable theory of Krasner, is proved. We do not claim to have a completely satisfactory generalization of Krasner's concept of a quasi-connected domain; in particular, while it is not sufficient only to assume that a subset of $\mathfrak{K}^n$ be axially quasi-connected, it seems as though our definition of W in the statement of the theorem is overly restrictive. However, it is only regions so defined with which we will be concerned in [**4**].

It is necessary to introduce some new ideas before the uniqueness theorem is stated.

DEFINITION 2.1. Let $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$, $\eta = (\eta_1, \eta_2, \cdots, \eta_n)$ be a pair of elements of $\mathfrak{K}^n$; we say that ξ is *directly axially joined* to η if $\xi_i = \eta_i$ for all but possibly one of the indices $i = 1, 2, \cdots, n$. If U is a subset of $\mathfrak{K}^n$, and if ξ, η are elements of U, we say that ξ and η are *U-axially joined* if there is a sequence $\eta = \xi^{(0)}, \xi^{(1)}, \cdots, \xi^{(N)} = \xi$ with the property that, for $i = 0, 1, 2, \cdots, N$, $\xi^{(i)} \in U$, and, for $i = 1, 2, \cdots, N$, $\xi^{(i-1)}$ is directly axially joined to $\xi^{(i)}$.

It is clear from the definition that "is U-axially joined to" is an equivalence relation.

DEFINITION 2.2. For $U \subset W \subset \mathfrak{K}^n$, we define the *axial join of U in W*, W', by

$$W' = \{\xi \in W : \xi \text{ is } W\text{-axially joined to an element of } U\}.$$

PROPOSITION 2.3. *If* $R(X_1, X_2, \cdots, X_n) \in \mathfrak{O}[X_1, X_2, \cdots, X_n]$, $\text{ord } R(0, 0, \cdots, 0) = 0$, *let* $W = \{(\xi_1, \cdots, \xi_n) \in \mathfrak{O}^n : \text{ord } R(\xi_1, \cdots, \xi_n) = 0\}$ *and let* $\rho_1, \rho_2, \cdots, \rho_n$ *be a set of positive numbers such that* $U = \Gamma_{\rho_1} \times \Gamma_{\rho_2} \times \cdots \times \Gamma_{\rho_n}$ *is not empty. Then, if* W' *denotes the axial join of* U *in* W, $W' = W$.

PROOF. If $n = 1$, any two elements of $\mathfrak{K}$ are directly axially joined, and so $W' = W$ trivially.

Assume the validity of the proposition for polynomials in $n-1$ variables with coefficients in $\mathfrak{O}$, $n \geqq 2$, and let $(\psi_1, \cdots, \psi_n) \in W$. We shall construct an element $(\eta_1, \eta_2, \cdots, \eta_n) \in U$ which is W-axially joined to $(\psi_1, \psi_2, \cdots, \psi_n)$.

Consider the image $\bar{R}(X_1, \cdots, X_n)$ of $R(X_1, \cdots, X_n)$ under the residue class map: it follows from the definition of W that, if $\bar{\xi}_i$ denotes the residue class of ξ_i under reduction mod $\mathfrak{P}$, $(\xi_1, \cdots, \xi_n)$ is an element of W if and only if $\bar{R}(\bar{\xi}_1, \cdots, \bar{\xi}_n) \neq 0$. Let the polynomials $\bar{R}'$, $\bar{R}'_0$ in $k[X_n]$ be defined by

$$\bar{R}'_0(X_n) = \bar{R}(0, 0, \cdots, 0, X_n), \qquad \bar{R}'(X_n) = \bar{R}(\bar{\psi}_1, \bar{\psi}_2, \cdots, \bar{\psi}_{n-1}, X_n);$$

since $\bar{R}'_0(0)\bar{R}'(\bar{\psi}_n) \neq 0$, the product of these two polynomials is not the zero polynomial. But k is infinite, so the existence of an element η of $\mathfrak{O}$ with the property $\bar{R}'(\bar{\eta})\bar{R}'_0(\bar{\eta}) \neq 0$ is guaranteed.

Let $R^*(X_1, X_2, \cdots, X_{n-1}) = R(X_1, X_2, \cdots, X_{n-1}, \eta)$ and put $W^* = \{(\xi_1, \xi_2, \cdots, \xi_{n-1}) \in \mathfrak{O}^{n-1}: \text{ord } R^*(\xi_1, \xi_2, \cdots, \xi_{n-1}) = 0\}$; then ord $R^*(0, 0, \cdots, 0) = 0$ and $(\psi_1, \psi_2, \cdots, \psi_{n-1}) \in W^*$, and therefore, by the induction hypothesis, $(\psi_1, \psi_2, \cdots, \psi_{n-1})$ is W^*-axially joined to an element $(\eta_1, \eta_2, \cdots, \eta_{n-1}) \in \Gamma_{\rho_1} \times \Gamma_{\rho_2} \times \cdots \times \Gamma_{\rho_{n-1}}$. Thus, if we choose any element η_n of Γ_{ρ_n}, the conclusion follows from the fact that $(\psi_1, \psi_2, \cdots, \psi_{n-1}, \eta)$ and $(\psi_1, \psi_2, \cdots, \psi_n)$ are directly axially joined, $(\eta_1, \eta_2, \cdots, \eta_{n-1}, \eta)$ and $(\eta_1, \eta_2, \cdots, \eta_n)$ are axially joined, and $(\xi_1, \xi_2, \cdots, \xi_{n-1}) \in W^*$ if and only if $(\xi_1, \xi_2, \cdots, \xi_{n-1}, \eta) \in W$.

REMARK 2.4. If $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$, $\eta = (\eta_1, \eta_2, \cdots, \eta_n)$ are elements of $\mathfrak{O}^n$, and if $\bar{\xi}_i = \bar{\eta}_i$, $i = 1, 2, \cdots, n$, then $\xi \in W$ if and only if $\eta \in W$.

THEOREM 2.5. *For $R(X_1, X_2, \cdots, X_n) \in \mathfrak{O}[X_1, X_2, \cdots, X_n]$, $R(0, 0, \cdots, 0) \in \Gamma_0$, let $W = \{(\xi_1, \cdots, \xi_n) \in \mathfrak{O}^n: \text{ord } R(\xi_1, \cdots, \xi_n) = 0\}$, and let $\{f_m(X_1, X_2, \cdots, X_n)\}$, $\{g_m(X_1, X_2, \cdots, X_n)\}$, $m = 1, 2, 3, \cdots$, be sequences of rational functions defined on W and converging uniformly to functions $f(X_1, X_2, \cdots, X_n)$ and $g(X_1, X_2, \cdots, X_n)$, respectively, on W. Suppose, for some set of positive numbers $\rho_1, \rho_2, \cdots, \rho_n$, the set $U = \Gamma_{\rho_1} \times \Gamma_{\rho_2} \times \cdots \times \Gamma_{\rho_n}$ is not empty and $f(X_1, X_2, \cdots, X_n) = g(X_1, X_2, \cdots, X_n)$ on U. Then $f(X_1, X_2, \cdots, X_n) = g(X_1, X_2, \cdots, X_n)$ identically on W.*

PROOF. Let ξ be any element of W. Then, by Proposition 2.3, there is an element η of U and a sequence $\eta = \xi^{(0)}, \xi^{(1)}, \cdots, \xi^{(N)} = \xi$ of elements of W such that adjacent members of the sequence are directly axially joined. The theorem will follow from construction of a sequence

$\Xi^{(0)}, \Xi^{(1)}, \cdots, \Xi^{(N)}$ of sets, each of which satisfies the conditions:

(1) $\Xi^{(i)} \subset W$.

(2) For any choice of j, $1 \leqq j \leqq n$, and any element $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$ of $\Xi^{(i)}$, the set

$$\Xi^{(i)}_{\xi,j} = \{\hat{\xi}_j : (\xi_1, \xi_2, \cdots, \xi_{j-1}, \hat{\xi}_j, \xi_{j+1}, \cdots, \xi_n) \in \Xi^{(i)}\}$$

has ξ_j as a limit point.

(3) $\xi^{(i)} \in \Xi^{(i)}$.

(4) If $(\xi_1, \cdots, \xi_n) \in \Xi^{(i)}$, then $f(\xi_1, \cdots, \xi_n) = g(\xi_1, \cdots, \xi_n)$.

Such a sequence is constructed inductively. If the initial member $\Xi^{(0)}$ is set equal to U, it follows from the hypothesis that $\Xi^{(0)}$ satisfies the four conditions.

Now, let $0 \leqq i < N$, and suppose the set $\Xi^{(i)}$ has been chosen in such a manner that conditions (1)–(4) are satisfied. Let

$$\Xi^{(i+1)} = \bigcup_{j=1}^{n} \{(\xi_1, \xi_2, \cdots, \xi_{j-1}, \eta_j, \xi_{j+1}, \cdots, \xi_n) \in W: \text{ for some } \xi_j,\ (\xi_1, \cdots, \xi_j, \cdots, \xi_n) \in \Xi^{(i)}\}.$$

It is obvious that $\Xi^{(i+1)}$ so defined satisfies the first of our conditions.

Suppose $\psi = (\psi_1, \cdots, \psi_n) \in \Xi^{(i+1)}$; if j is any integer, $1 \leqq j \leqq n$, we must show that $\Xi^{(i+1)}_{\psi,j}$ has ψ_j as a limit point. But $\psi \in \Xi^{(i+1)}$ implies the existence of an integer j', $1 \leqq j' \leqq n$, and an element $\xi = (\xi_1, \cdots, \xi_n) \in \Xi^{(i)}$ such that $\psi_i = \xi_i$ if $i \neq j'$. If $j' = j$, Remark 2.4 implies that all elements congruent to ψ_j mod $\mathfrak{P}$ are in $\Xi^{(i+1)}_{\psi,j}$, and so ψ_j is certainly a limit point of this latter set. On the other hand, if $j' \neq j$, we use the fact that ξ_j is a limit point of $\Xi^{(i)}_{\xi,j}$. Thus, we can choose an infinite subset $\{\hat{\xi}_{j,l}\}$, $l = 0,1,2, \cdots$, of $\Xi^{(i)}_{\xi,j}$ such that $\hat{\xi}_{j,l} \to \xi_j$ as $l \to \infty$, and such that these elements are all in the same residue class mod $\mathfrak{P}$; but then, if we define $\phi_l = (\phi_{l1}, \phi_{l2}, \cdots, \phi_{ln})$ by

$$\begin{aligned} \phi_{li} &= \psi_{j'} && \text{if } i = j', \\ &= \hat{\xi}_{j,l} && \text{if } i = j, \\ &= \xi_i && \text{otherwise,} \end{aligned}$$

Remark 2.4 implies that $\phi_l \in \Xi^{(i+1)}$ for all l, and, as $l \to \infty$, $\phi_{lj} \to \xi_j = \psi_j$, from which it follows that condition (2) is satisfied by the set $\Xi^{(i+1)}$.

Condition (3) is fulfilled since $\xi^{(i)} \in \Xi^{(i)}$ and $\xi^{(i)}$, $\xi^{(i+1)}$ are directly axially joined.

Finally, if $\psi = (\psi_1, \cdots, \psi_n) \in \Xi^{(i+1)}$, we choose $\xi = (\xi_1, \cdots, \xi_n)$ and the integer j such that $\psi_i = \xi_i$ if $i \neq j$. Let

$$R_{\psi,j}(X) = R(\psi_1, \psi_2, \cdots, \psi_{j-1}, X, \psi_{j+1}, \cdots, \psi_n),$$

and let $W_{\psi,j} = \{\eta_j : \operatorname{ord} R_{\psi,j}(\eta_j) = 0\}$. Proposition 2.3 tells us that $W_{\psi,j}$ is a quasi-connected domain; but, by the induction hypothesis,

$f(\xi_1, \cdots, \xi_{j-1}, X, \xi_{j+1}, \cdots, \xi_n)$ is identically equal to

$$g(\xi_1, \cdots, \xi_{j-1}, X, \xi_{j+1}, \cdots, \xi_n)$$

on the set $\Xi^{(i)}_{\xi,j}$; since this subset of $W_{\psi,j}$ has a limit point in itself, application of the one-variable uniqueness theorem proved by Krasner gives the result that

$$f(\xi_1, \cdots, \xi_{j-1}, X, \xi_{j+1}, \cdots, \xi_n) = g(\xi_1, \cdots, \xi_{j-1}, X, \xi_{j+1}, \cdots, \xi_n)$$

identically on $W_{\psi,j}$. This proves our theorem.

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