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BOUNDED PROJECTIONS ON FOURIER-STIELTJES TRANSFORMS

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ABSTRACT. We study certain algebraic projections on the measure algebra (of a locally compact abelian group) which extend to bounded projections on the uniform closure of the Fourier-Stieltjes transforms. These projections arise by studying a Raikov system of subsets induced by locally compact subgroups. These results generalize the inequality $\|\hat{\mu}_d\|_\infty \leqq \|\hat{\mu}\|_\infty$ (where μ is in the measure algebra, μ_d is the discrete part of μ, and $\|\hat{\mu}\|_\infty$ is the sup-norm of the Fourier-Stieltjes transform).

Here H will be a locally compact abelian (LCA) group. The group H with the discrete topology is denoted H_d. This is the same as giving H the topology induced from declaring the subgroup $G = \{0\} \subset H$ to be open. The space of finite regular Borel measures on H is denoted $M(H)$. For $\mu \in M(H)$, let μ_d denote the discrete part of μ. The ring homomorphism $\mu \mapsto \mu_d$ maps $M(H)$ onto $M(H_d)$, and this map is norm-nonincreasing in the measure norm; that is, $\|\mu_d\| \leqq \|\mu\|$, $\mu \in M(H)$. For $\mu \in M(H)$, we let $\hat{\mu}$ denote the Fourier-Stieltjes transform of μ; that is

$$\hat{\mu}(\gamma) = \int_H (\gamma(x))^- \, d\mu(x),$$

$\gamma \in \hat{H}$ (the dual of H). In two previous papers **[2]**, **[3]**, we showed (in a more general setting) $\|\hat{\mu}_d\|_\infty \leqq \|\hat{\mu}\|_\infty$, $\mu \in M(H)$ (where $\|\cdot\|_\infty$ denotes the sup-norm). This further implies that $\mathcal{M}(\hat{H}) = \mathcal{M}_c(\hat{H}) \oplus \mathcal{M}_d(\hat{H})$, where $\mathcal{M}(\hat{H})$, $\mathcal{M}_c(\hat{H})$, and $\mathcal{M}_d(\hat{H})$ are the sup-norm closures on $\hat{H}$ of the Fourier-Stieltjes transforms of measures from $M(H)$, $M_c(H)$ (the space of continuous measures), and $M(H_d)$ respectively. Let Δ denote the maximal ideal space of $M(H)$, and let $\kappa\hat{H}$ denote the Δ-closure of $\hat{H}$ in Δ. (Recall $\hat{H} \subset \Delta$ under the identification map from $\hat{H}$ to Δ by $\pi_\gamma(\mu) = \hat{\mu}(\gamma)$, $\gamma \in \hat{H}$, $\mu \in M(H)$.) We call the set $\kappa\hat{H} \backslash \hat{H}$ the fringe of $\hat{H}$. The result

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$\|\hat{\mu}_d\|_\infty \leqq \|\hat{\mu}\|_\infty$ ($\mu \in M(H)$) implies that the fringe of $\hat{H}$ contains a homeomorphic copy of the Bohr group $\beta\hat{H}$ of $\hat{H}$ (under the map $\chi \mapsto \pi_\chi$ from $\beta\hat{H}$ to Δ given by $\pi_\chi(\mu) = \int_H \bar{\chi}\, d\mu_d$, $\mu \in M(G)$, $\chi \in \beta\hat{H}$).

The setting in this paper is as follows. We let H be an LCA group with topology $\mathcal{T}_H$, and G a subgroup of H which has an LCA group topology $\mathcal{T}_G$ such that the injection $(G, \mathcal{T}_G) \to (H, \mathcal{T}_H)$ is continuous. For example, suppose G is the image under a continuous monomorphism of an LCA group. We let H_G denote H with the topology induced by declaring the subgroup G with the $\mathcal{T}_G$-topology to be open. We will assume that G is a *nonopen* subgroup of H so that $H \neq H_G$ topologically.

We now will define the natural projection $P: M(H) \to M(H_G)$ by utilizing a Raikov system of subsets of H. (For the basic facts concerning Raikov systems see [**5**].) Let $\mathcal{F}$ denote a family of σ-compact subsets of H such that: (1) if $A \in \mathcal{F}$, B is σ-compact, and $B \subset A$, then $B \in \mathcal{F}$, (2) if $\{A_n\}_{n=1}^\infty \subset \mathcal{F}$, then $\bigcup_{n=1}^\infty A_n \in \mathcal{F}$, (3) if $A, B \in \mathcal{F}$, then $A + B \in \mathcal{F}$, and (4) if $A \in \mathcal{F}$ and $x \in H$, then $x + A \in \mathcal{F}$. Such a family of subsets of H is called a Raikov system. We choose $\mathcal{F}$ to be the Raikov system generated by the family of compact subsets of G.

Let R be the set of measures $\mu \in M(H)$ such that $|\mu|$ is concentrated on some elements of $\mathcal{F}$, and let I be the set of measures $\mu \in M(H)$ such that $|\mu|\,(A) = 0$ for all $A \in \mathcal{F}$. Then I is a closed ideal in $M(H)$ and R is a closed subalgebra of $M(H)$. Furthermore, $M(H) = R \oplus I$ (see, for example, [**5**, p. 151]). Now R can be identified with $M(H_G)$, and thus the natural projection $P: M(H) \to M(H_G)$ is induced by the given direct sum. For $\mu \in M(H)$, we write $\mu = \mu_G + \mu_I$ where $\mu_G \in M(H_G)$ and $\mu_I \in I$. Thus $P\mu = \mu_G$, $\mu \in M(H)$. Observe that P is a norm-bounded projection; that is, $\|P\mu\| \leqq \|\mu\|$, $\mu \in M(H)$. Our goal now is to show

$$\|(P\mu)^\wedge\|_\infty \leqq \|\hat{\mu}\|_\infty, \quad \mu \in M(H).$$

Let $\phi: H_G \to H$ be the identity map and $\hat{\phi}: \hat{H} \to \hat{H}_G$ the adjoint map (an injection). In an earlier paper [**4**], we showed for any continuous homomorphism $\pi: G_1 \to G_2$ (G_1, G_2 LCA groups) that π is open if and only if $\hat{\pi}: \hat{G}_2 \to \hat{G}_1$ (the adjoint map) is proper (the inverse image of a compact set is compact). Thus since ϕ is not open, $\hat{\phi}$ is not proper. The map ϕ induces a continuous homomorphism $\phi^*: M(H_G) \to M(H)$. Since ϕ is one-to-one, $\hat{\phi}\hat{H}$ is dense in $\hat{H}_G$. Indeed for any compact $K \subset \hat{H}$, $\hat{\phi}(\hat{H} \backslash K)$ is dense in $\hat{H}_G$. For $\mu \in M(H_G)$, $\|\hat{\mu}\|_\infty$ is the supremum of $|\hat{\mu}|$ over either $\hat{\phi}\hat{H}$ or $\hat{H}_G$. (We will identify $\hat{\phi}\hat{H}$ and $\hat{H}$ as subsets of $\hat{H}_G$ when convenient.)

For an LCA group L, we let $P(L)$ denote the space of continuous positive definite functions on L; we let $P_c(L)$ be those $f \in P(L)$ with compact support.

We will denote the Haar measure on H_G by λ. (The measure λ restricted to G is the Haar measure on G.)

PROPOSITION 1. *Let $f \in P_c(H_G)$ and let $d\mu = f\,d\lambda$. If $g \in P_c(H)$, then $g * \mu$ (convolution in $M(H)$) is in $P_c(H)$.*

PROOF. Since $f \in P_c(H_G)$, $\hat{f} \in L^1(\hat{H}_G)$ by the inversion theorem [7, p. 22], and $\hat{f} \geqq 0$ by Bochner's theorem [7, p. 19]. Thus for $\gamma \in \hat{H} \subset \hat{H}_G$, $\hat{\mu}(\gamma) = \int_H \bar{\gamma}\, d\mu = \int_{H_G} \bar{\gamma} f\, d\lambda = \hat{f}(\gamma) \geqq 0$.

Since g and μ have compact supports, $g * \mu$ is a continuous function on H with compact support. Finally, $g * \mu$ is positive definite since $(g * \mu)\hat{} = \hat{g}\hat{\mu} \geqq 0$ on $\hat{H}$. □

An LCA group L is amenable, and thus satisfies the condition of Godement: the constant function 1 can be approximated uniformly on compact subsets of L by functions of the form $k * \tilde{k}$, where k is a continuous function with compact support and $\tilde{k}(x) = (k(-x))^-$, $x \in L$. (See [6, p. 168, 172].) Thus we have:

PROPOSITION 2. *Let L be an LCA group and $K \subset L$ a compact subset of L. Given $\varepsilon > 0$, there is $p \in P_c(L)$ such that $p(0) = 1$ and $|p - 1| < \varepsilon$ on K.*

PROPOSITION 3. *Let K be a compact subset of H_G, and let U be a relatively compact neighborhood of 0 in H_G. Then there is a neighborhood V of 0 in H such that $(x + V) \cap K \subset x + U$ for all $x \in K$.*

PROOF. Since K is compact in H_G, $K - K$ is also compact in H_G; and the induced topology on $K - K$ from H agrees with the H_G-topology on $K - K$ (since compact topologies are minimal Hausdorff). Thus there is an H-open neighborhood of 0, V, such that $V \cap (K - K) \subset U \cap (K - K)$. Thus for $x \in K$, $(x + V) \cap K \subset x + (V \cap (K - \{x\})) \subset x + (V \cap (K - K)) \subset x + (U \cap (K - K)) \subset x + U$. □

PROPOSITION 4. *Let $\xi \in \hat{H}_G$, K a compact subset of H_G, and $\varepsilon > 0$ be given. Then there exists $\gamma \in \hat{H}$ such that $|\gamma - \xi| < \varepsilon$ on K.*

PROOF. Recall that $\hat{\phi}\hat{H}$ can be identified with $\hat{H}$, and it is dense in $\hat{H}_G$. Finally, the topology in $\hat{H}_G$ is the compact-open topology. □

THEOREM 5. *Let $P: M(H) \to M(H_G)$. Then $\|(P\mu)\hat{}\|_\infty \leqq \|\hat{\mu}\|_\infty$, $\mu \in M(H)$.*

PROOF. Let $\mu \neq 0$ be in $M(H)$, and let $\xi \in \hat{H}_G$. Write $\mu = \mu_G + \mu_I$ where $\mu_G \in M(H_G)$ and $\mu_I \in I$ using the previously described Raikov system. We will show $|\hat{\mu}_G(\xi)| \leqq \|\hat{\mu}\|_\infty$.

We may assume spt μ_G (spt denotes the support) is compact in H_G. By Proposition 2, there is $p \in P_c(H_G)$ such that $p(0) = 1$ and $|p - 1| < \varepsilon/\|\mu\|$ on spt μ_G.

Since $|\mu_I|$ (spt p) $= 0$, we may assume spt $\mu_I \cap$ spt $p = \varnothing$. Since p is uniformly continuous in the H_G-topology, there is a H_G-open neighborhood of 0, U, such that for $x \in H_G$ and $y \in U$, $|p(x + y) - p(x)| < \varepsilon/\|\mu\|$. Let $K = -K$ be a compact subset of H_G containing spt p and spt μ_G. By Proposition 3, choose V to be an H-open neighborhood of 0 such that $V = -V$ and $(x + V) \cap K \subset x + U$ for all $x \in K$; we further assume that (spt $p + V$) $\cap$ (spt $\mu_I + V$) $= \varnothing$.

Now choose $\gamma \in \hat{H}$ by Proposition 4 such that $|\gamma - \bar{\xi}| < \varepsilon/\|\mu\|$ on K; and choose $g \in P_c(H)$ with spt $g \subset V$, $g \geqq 0$, and $\int_U g \, d\lambda = 1$. For any $x \in K$, $|(g * p \, d\lambda)(x) - p(x)| = |\int_V g(y)p(x - y) \, d\lambda(y) - p(x)| = |\int_U g(y)(p(x - y) - p(x)) \, d\lambda(y)| < \varepsilon/\|\mu\|$ (since $V \cap (x - K) \subset U$, $x \in$ spt p). Thus letting $f = g * p \, d\lambda$, spt $f \subset V +$ spt p and $f \in P_c(H)$ (by Proposition 1). Also $f(0) < p(0) + \varepsilon/\|\mu\| = 1 + \varepsilon/\|\mu\|$, and spt $f \cap$ spt $\mu_I = \varnothing$. For $x \in$ spt μ_G,

$$|f(x) - 1| \leqq |f(x) - p(x)| + |p(x) - 1| < 2\varepsilon/\|\mu\|.$$

And

$$\begin{aligned}\left|\int_{H_G} \bar{\xi} \, d\mu_G - \int_H \gamma f \, d\mu\right| &\leqq \left|\int_{H_G} \bar{\xi} \, d\mu_G - \int_{H_G} \gamma \, d\mu_G\right| \\ &\quad + \left|\int_{H_G} \gamma \, d\mu_G - \int_H \gamma f \, d\mu_G\right| + \left|\int_H \gamma f \, d\mu_I\right| \\ &< (\varepsilon/\|\mu\|) \, \|\mu_G\| + (2\varepsilon/\|\mu\|) \, \|\mu_G\| + 0 \\ &\leqq 3\varepsilon.\end{aligned}$$

Now $|\int_H \gamma f \, d\mu| \leqq f(0) \, \|\hat{\mu}\|_\infty < (1 + \varepsilon/\|\mu\|) \, \|\hat{\mu}\|_\infty$ (since γf is positive definite).

Summarizing, given $\xi \in \hat{H}_G$,

$$\begin{aligned}|\hat{\mu}_G(\xi)| = \left|\int_{H_G} \bar{\xi} \, d\mu_G\right| &\leqq \left|\int_H \gamma f \, d\mu\right| + 3\varepsilon \\ &\leqq (1 + \varepsilon/\|\mu\|) \, \|\hat{\mu}\|_\infty + 3\varepsilon \leqq \|\hat{\mu}\|_\infty + 4\varepsilon.\end{aligned}$$

And so $\|\hat{\mu}_G\|_\infty \leqq \|\hat{\mu}\|_\infty$. □

COROLLARY 6. *Let $\mathcal{M}(\hat{H})$, $\mathcal{M}(\hat{H}_G)$ and $\mathfrak{I}$ denote the uniform closures of the Fourier-Stieltjes transforms of $M(H)$, $M(H_G)$, and I respectively. Then $\mathcal{M}(\hat{H}) = \mathcal{M}(\hat{H}_G) \oplus \mathfrak{I}$.*

COROLLARY 7. *If $\mu \in M(H)$ and $\hat{\mu} \in \mathcal{M}(\hat{H}_G)$, then $\mu \in M(H_G)$.*

COROLLARY 8. *Let $\hat{H}_G$ be embedded in $\kappa\hat{H}$ (the maximal ideal space of $\mathcal{M}(\hat{H})$; equivalently, the closure of $\hat{H}$ in Δ), by $\gamma \mapsto \pi_\gamma$ from $\hat{H}_G$ to $\kappa\hat{H}$ where $\pi_\gamma(\mu) = \hat{\mu}_G(\gamma)$ ($\mu \in M(H)$). Since $\pi_\gamma(\mu) = 0$ for $\mu \in L^1(H)$ (recall G is nonopen in H), $\pi_\gamma \in \kappa\hat{H}\backslash\hat{H}$ (the fringe of $\hat{H}$). In particular, for $\mu \in M(H)$, $\|\hat{\mu}_G\|_\infty = \lim\sup \|\hat{\mu}_G\|_\infty \leqq \|\hat{\mu}\|_\infty$.*

These corollaries follow from the inequality $\|\hat{\mu}_G\|_\infty \leqq \|\hat{\mu}\|$ ($\mu \in M(H)$). The proofs are discussed in a more general setting in **[3]**. For $G = \{0\}$ (and thus $H_G = H_d$), Corollary 7 is due to E. Hewitt for H with a restricted hypothesis and to W. Eberlein in general. A reference for these facts, plus a different (although closely related) direct sum decomposition, is **[1]**.

Some interesting examples of LCA groups H with a nonopen subgroup G are: (1) H nondiscrete and $G = \{0\}$, (2) G noncompact and $H = \beta G$ the Bohr compactification of G, (3) $G = R$ (the real numbers) and H a compact solenoidal group, and (4) certain local direct product groups embedded in the appropriate complete direct product groups.

BIBLIOGRAPHY

1. R. Burckel, *Weakly almost periodic functions on semigroups*, Gordon and Breach, New York, 1970.

2. C. Dunkl and D. Ramirez, *C*-algebras generated by measures*, Bull. Amer. Math. Soc. **77** (1971), 411–412.

3. ———, *C*-algebras generated by Fourier-Stieltjes transforms*, Trans. Amer. Math. Soc. (to appear).

4. ———, *Homomorphisms on groups and induced maps on certain algebras of measures*, Trans. Amer. Math. Soc. **160** (1971), 475–485.

5. E. Hewitt, *The asymmetry of certain algebras of Fourier-Stieltjes transforms*, Michigan Math. J. **5** (1958), 149–158. MR **21** #4993.

6. H. Reiter, *Classical harmonic analysis and locally compact groups*, Oxford Math. Monographs, Clarendon Press, Oxford, 1968.

7. W. Rudin, *Fourier analysis on groups*, Interscience Tracts in Pure and Appl. Math., no. 12, Interscience, New York, 1962. MR **27** #2808.

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