CONTINUITY AND LINEARITY OF CENTRALIZERS ON A COMPLEMENTED ALGEBRA[1]

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ABSTRACT. Let A be a semisimple complemented algebra and let T be a mapping of A into itself such that either $T(xy) = xTy$ or $T(xy) = (Tx)y$ holds for all $x, y \in A$. If T is defined everywhere on A then T is a bounded linear operator.

1. A right centralizer on an algebra A is a mapping T of A into A such that $T(xy) = xTy$ for all $x, y \in A$. A left centralizer is a mapping $T: A \to A$ such that $T(xy) = (Tx)y$ for all $x, y \in A$. This terminology is somewhat different from the terminology of [5] and is due to B. Johnson, who developed the theory of centralizers in [3] and was able to show that for a certain class of Banach algebras centralizers are always linear and bounded. In [4] he showed that this is the case when the algebra has a certain type of bounded approximate identity.

The purpose of this paper is to extend these results of Johnson to the case of complemented algebras. Results of [4] are not applicable to our case. In fact the authors are convinced that Johnson's condition on existence of a certain type of approximate identity in the case of complemented algebras would be equivalent to the assumption of finite-dimensionality of the algebra.

Inasmuch as every proper right H^*-algebra [8] is a complemented algebra, we have extension of Johnson's result to all types of H^*-algebras, and in particular, to the algebra of Hilbert Schmidt operators.

We developed our theory for right centralizers but it is obvious that the same theory could be developed for left centralizers.

2. In this section we recall some basic definitions and facts from the theory of complemented algebras. For a more complete background the reader is referred to [6] and [7].

A complemented algebra is a Banach algebra A with a Hilbert space norm such that the orthogonal complement of a right (left) ideal is again a right (left) ideal. We say that $x^l \in A$ is a left adjoint of $x \in A$ if $(xy, z) = (y, x^l z)$ holds for all $y, z \in A$. A left projection is a nonzero member e of A such that $e^2 = e^l = e$. Note that the inequality $\|ex\| \leq \|x\|$ holds for each $x \in A$ and any left projection e in A (in this case $(ex, x - ex) = 0$ and so $\|x\|^2 = \|ex\|^2 + \|x - ex\|^2$). Projections e_1, e_2 are said to be doubly orthogonal if $e_1 e_2 = e_2 e_1 = 0$. A primitive left projection is a left projection which cannot be written as a sum of two doubly orthogonal left projections. Each left projection can be written as a sum of a finite number of mutually doubly orthogonal primitive left projections. The right quasi-inverse of $a \in A$ is a member b of A such that $a + b + ab = 0$ [2]. If a Banach algebra is semisimple then each nonzero ideal contains an element which has no right quasi-inverse [2]. A semisimple complemented algebra A has the property that $Ax = 0$ implies $x = 0$, i.e. in the terminology of Johnson a semisimple complemented algebra is faithful. If e is a primitive left projection and A_0 is the smallest closed two-sided ideal containing e then A_0 is a simple algebra.

3. We will need a few lemmas from the theory of complemented algebras. Let A be a fixed semisimple complemented algebra.

LEMMA 1.   *Let e_1, e_2 be primitive left projections belonging to the same minimal closed two-sided ideal A_0 in A. Then the subspaces $e_i A e_j$ are one-dimensional and the subspaces $e_1 A e_1$, $e_2 A e_2$ are isomorphic to the complex field. Also there are elements $e_{ij} \in a_i A e_j$ such that $e_{ii} = e_i$, $e_{ij}^l = e_{ji}$ and $e_{ij} e_{jk} = e_{ik}$ for all i, j, k in the set $\{1, 2\}$.*

PROOF.   First part of the lemma follows from [6, Lemma 7] and the fact that $e_i A e_j = e_i A_0 e_j$. The matrix units e_{ij} can be selected as follows: first set $e_{11} = e_1$, $e_{22} = e_2$ and take $e_{12} \in e_1 A e_2$ so that $\|e_{12}\| = \|e_2\|$; then set $e_{21} = e_{12}^l$. The equalities $e_{ij} e_{jk} = e_{ik}$ are then easily verified (compare with pp. 381, 382 of [1]).

LEMMA 2.   *Each nonzero left ideal L in A contains a left projection.*

PROOF.   We use the technique of [6]. Let a be a member of L which has no right quasi-inverse. Then $-a$ is a relative identity of the proper right ideal $R =$ closure of $\{ax + x \mid x \in A\}$. Write $-a = e + u$ with $e \in R^p$, $u \in R$. Then [6, Lemma 2], e is a left projection and $eu = 0$. It follows that $e = e(e + u) = e(-a)$ is a member of L.

COROLLARY.   *Every left ideal in A contains a primitive left projection.*

LEMMA 3.   *There exists a family $\{e_\alpha\}_{\alpha \in \Gamma}$ of primitive left projections such that $A = \sum_{\alpha \in \Gamma} A e_\alpha$ and $(Ae_\alpha, Ae_\beta) = 0$ if $\alpha \neq \beta$ for some $\alpha, \beta \in \Gamma$ (members of $\{e_\alpha\}_{\alpha \in \Gamma}$ do not have to be doubly orthogonal).*

PROOF. It is sufficient to consider a maximal family $\{e_\alpha\}$ of primitive left projections such that $(Ae_\alpha, Ae_\beta) = 0$ if $\alpha \neq \beta$.

COROLLARY. *If $x \in A$ is such that $ex = 0$ for each primitive left projection e in A then $x = 0$.*

LEMMA 4. *Let e_1 be a primitive left projection in A and let $a \in e_1 A$. Then there exists a left projection $f \in A$ such that $af = a$.*

PROOF. Let A_0 be the smallest closed two-sided ideal containing e and let f be a left projection in the left ideal $Aa = Ae_1 a = A_0 e_1 a$. Then $f = xe_1 a$ for some $x \in A$ and there is a primitive left projection e_2 in A_0 such that $e_2 f \neq 0$. Take e_{12} as in Lemma 1 above. Then $e_{12} f = e_{12} x e_1 a = \lambda e_1 a$ for some nonzero complex number λ (since $e_1 A e_1$ is isomorphic to the complex field). Then $a = e_1 a = \lambda^{-1} e_{12} f$ and so $af = a$.

LEMMA 5. *If $a, b \in eA$ for some primitive left projection e then there exists a left projection f such that both $af = a$ and $bf = b$.*

PROOF. Let f_1, f_2 be left projections such that $af_1 = a$, $bf_2 = b$. Let $e_1, \cdots, e_n, e_{n+1}, \cdots, e_m$ be primitive left projections such that $f_1 = \sum_{i=1}^n e_i$, $f_2 = \sum_{i=n+1}^m e_i$. Then the subspaces $e_i A e_j$ are one dimensional and so the algebra A' generated by elements of the form $e_i x e_j$, $x \in A$, is a finite-dimensional left H^*-algebra (one can define a left H^*-algebra in the same way that a right H^*-algebra was defined in [8]). It is then easy to see that A' has an identity f and $f^l = f$. It follows then that $af = af_1 f = af_1 = a$ and $bf = bf_2 f = bf_2 = b$.

4. Now let T be a right centralizer defined everywhere on A. Then T is homogeneous [3, Lemma 1, p. 307], i.e. $T(\lambda x) = \lambda Tx$ for all $x \in A$ and each complex number λ. Our aim now is to show that T is a bounded linear operator. This will be done through several lemmas.

LEMMA 6. *T is linear on eA for any primitive left projection e in A.*

PROOF. We need only to establish the additivity. Let $x, y \in eA$ and let f be a left projection such that $xf = x$, $yf = y$. Then $T(x + y) = T((x + y)f) = (x + y)Tf = xTf + yTf = T(xf) + T(yf) = Tx + Ty$.

LEMMA 7. *If a complemented algebra A has a left identity i then A is finite dimensional. The same is true if we assume that A has a right identity.*

PROOF. First note that the left identity i is a left projection and so $i = \sum_{i=1}^n e_i$ for some mutually doubly orthogonal primitive left projections $e_1, \cdots, e_n$ [6, p. 50]. Then $A = \sum_{i=1}^n e_i A$ and so it follows from Theorem 1 of [7] that A is a left H^*-algebra. But then $A = \sum_{ij} e_i A e_j$ is at most n^2-dimensional.

LEMMA 8. *Each infinite-dimensional semisimple complemented algebra A contains an infinite family of mutually doubly orthogonal (primitive) left projections.*

PROOF. Let Λ be a maximal family of mutually doubly orthogonal left projections. If Λ is finite, say $\Lambda = \{e_1, e_2, \cdots, e_n\}$, then $e = \sum_{i=1}^{n} e_i$ is also a left projection. It follows from Lemma 7 that there exists $a \in A$ such that $ae \neq a$. Then $b = ae - a$ belongs to the left annihilator $l(R)$ of the right ideal $R = \sum_{i=1}^{n} e_i A$ and so $l(R) \neq (0)$. Then as in the proof of Lemma 5 of [6] one can show that R^p contains a (nonzero) left projection. This would lead to a contradiction with the maximality of Λ.

Again let A be a semisimple complemented algebra and let T be as above.

LEMMA 9. *The restriction of T to eA is a bounded operator, where e is any primitive left projection in A.*

PROOF. Assume that T is not bounded on the unit sphere of eA. Then there exists a sequence $\{x_n\}$ of members of eA such that $\|x_n\| \leq 1$ and $\|Tx_n\| \geq n^2$ for all n. For each n let $y_n = n^{-1}x_n$. Then $\|y_n\| \leq 1/n$ and $\|Ty_n\| \to \infty$.

Now let A_0 be the smallest closed two-sided ideal containing e. Then $eA = eA_0$ is infinite dimensional (otherwise T would be bounded on eA) and so there exists a sequence $\{e_i\}$ of mutually doubly orthogonal primitive left projections such that $e_1 = e$. Also one can select as in [6, Theorem 3] a set $\{e_{ij}\}$ of members of $e_i A_0 e_j$ such that $e_{ii} = e_i$, $e_{ij}^l = e_{ji}$ and $e_{ij}e_{jk} = e_{ik}$ for all i, j, k. Then the series $\sum_{i=1}^{\infty} e_{i1}y_i$ converges to some member y of A_0 and we have $e_{1n}y = y_n$ for each n. It follows then that $\|Ty_n\|^2 = \|e_{1n}Ty\|^2 = (e_{1n}Ty, e_{1n}Ty) = (Ty, e_nTy) = \|e_nTy\|^2 \leq \|Ty\|$ which contradicts the fact that $\|Ty_n\| \to \infty$.

LEMMA 10. *T is additive on A.*

PROOF. If $x, y \in A$ and e is any minimal left projection then $eT(x + y) = T(ex + ey) = T(ex) + T(ey) = e(Tx + Ty)$ (because of Lemma 6 above). Thus $T(x + y) = Tx + Ty$.

COROLLARY. *T is linear on A.*

LEMMA 11. *T is bounded on A.*

PROOF. It is sufficient to show that the graph of T is closed. So let $\{x_n\} \subset A$ and $x, y \in A$ be such that $x_n \to x$ and $Tx_n \to u$. If e is any primitive left projection then $ex_n \to ex$ and $T(ex_n) = eTx_n \to eu$ and we can also conclude that $T(ex_n)$ tends to $T(ex) = eTx$ (since T is bounded on eA). Thus $eu = eTx$ for each primitive left projection e and this implies

that $u = Tx$. It follows from the Closed Graph Theorem that T is bounded.

Summing up and taking into account the fact that the same theory can be developed for left centralizers we state our main result.

THEOREM. *A right centralizer defined everywhere on a semisimple complemented algebra A is a bounded linear operator on A. The same is true for a left centralizer defined everywhere on A.*

ADDED IN PROOF.   It was pointed out to the authors that B. E. Johnson and A. M. Sinclair established continuity of an additive centralizer defined on a semisimple Banach algebra [Amer. J. Math. **90** (1968), 1068–1073]. In the present paper the authors do not assume additivity.

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