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ONE-TO-ONE MAPPINGS

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ABSTRACT. In an earlier paper, the author showed that E^2 can never be the image, under a nontopological 1-1 mapping, of a connected, locally connected, locally compact topological space. In this paper, we show that several other spaces, including S^2 and I^2 share this property with E^2.

By a *mapping* we will mean a continuous function. A topological space Y will be said to have *Property* H if for each connected, locally connected, locally compact topological space X and each 1-1 mapping f of X onto Y, f is a homeomorphism.

In [**1**] it was shown that the Euclidean plane has Property H. In the present paper, we show that several other spaces, including the 2-sphere and the 2-cell, also have this property.

We will let E^2, S^2, and I^2 denote, respectively, the plane, the unit 2-sphere, and the unit 2-cell. For an arbitrary 2-cell K, we will let Int K and Bd K denote, respectively, the interior and boundary of K, where K is regarded as a 2-manifold with boundary.

By a *generalized continuum*, we will mean a connected, locally compact metric space. (It follows from [**3**, Corollary, p. 111] that such a space is always separable.)

THEOREM 1. *Let Y be a locally connected, locally compact metric space having the following property: for each simple closed curve J in Y, there is a 2-cell K in Y such that $J =$* Bd *K and such that* Int *K is an open set in Y. Then Y has Property* H.

PROOF. Assume that Y does not have Property H. Then there is a connected, locally connected, locally compact topological space X and a nontopological 1-1 mapping f of X onto Y.

By [**2**, Theorem 1, p. 1321], X is metrizable and may therefore be regarded as a generalized continuum. It follows then from [**4**, 5.2, p. 38] that X is arcwise connected. Also, since X is connected and f is continuous,

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Y is connected and is, therefore, a locally connected generalized continuum. Hence, by [**5**, Theorem 7, p. 1430] there is a topological ray α in X such that $f(\alpha)$ is a simple closed curve in Y.

By our hypothesis, then, there is a 2-cell K in Y such that $f(\alpha) = \operatorname{Bd} K$ and such that $\operatorname{Int} K$ is an open set in Y. We shall show that $f^{-1}(K)$ is connected, locally connected, and locally compact.

Let $O = X - f^{-1}(K)$ and let $U = f^{-1}(\operatorname{Int} K)$. Then O and U are disjoint open sets in X and $f^{-1}(K) = \alpha \cup U$. To show that $f^{-1}(K)$ is connected, we observe that $X - \alpha = O \cup U$. Since each of X and α is connected, we have $\alpha \cup U$ $(= f^{-1}(K))$ connected. A similar argument shows that $f^{-1}(K)$ is locally connected. For if $x \in f^{-1}(K)$ and V is an open set in X with $x \in V$ then there is a connected open set W in X such that $x \in W \subset V$ and such that $W \cap \alpha$ is connected. Since $W \cap O$ and $W \cap U$ are disjoint open sets and $W - \alpha = (W \cap O) \cup (W \cap U)$, we have $(W \cap \alpha) \cup (W \cap U)$ connected; i.e., $W \cap f^{-1}(K)$ is a connected set. The local compactness of $f^{-1}(K)$ is an immediate consequence of the local compactness of X and the fact that $f^{-1}(K)$ is a closed subset of X. Thus, $f^{-1}(K)$ is connected, locally connected, and locally compact.

Now, letting $[0, 1)$ denote the real-line interval $\{t \mid 0 \leqq t < 1\}$, define X' and Y' to be subspaces of $X \times [0, 1)$ and $Y \times [0, 1)$, respectively, as follows:

$$X' = (f^{-1}(K) \times \{0\}) \cup (\alpha \times [0, 1))$$

and

$$Y' = (K \times \{0\}) \cup (\operatorname{Bd} K \times [0, 1)).$$

Then X' is connected, locally connected, and locally compact; and Y' is homeomorphic to E^2. Define a function g from X' onto Y' by letting $g(x, t) \doteq (f(x), t)$ for each $(x, t) \in X'$. Since f is 1-1 and continuous, g is 1-1 and continuous. Therefore, it follows from [**1**, Theorem 4.4, p. 308] that g is a homeomorphism. But this implies that $f \mid \alpha$ is a homeomorphism of α onto $\operatorname{Bd} K$. Since α is a topological ray and $\operatorname{Bd} K$ is a simple closed curve we have a contradiction and the proof is complete.

Each of the following corollaries is an immediate consequence of Theorem 1.

COROLLARY 1.1. *S^2 has Property* H.

COROLLARY 1.2. *I^2 has Property* H.

COROLLARY 1.3. *If Y is a simply connected, locally connected, locally compact subspace of E^2, then Y has Property* H.

It follows from Theorem 1 that if a topological space Y is the union of two 2-spheres which have exactly one point in common, then Y has Property H. Our next theorem gives us a more general result of this nature.

THEOREM 2. *Suppose that a topological space Y is the union of two closed subspaces Y_1 and Y_2 such that* (1) *Y_1 and Y_2 have exactly one point in common, and* (2) *each of Y_1 and Y_2 has Property* H. *Then Y has Property* H.

PROOF. Suppose that X is a connected, locally connected, locally compact topological space and that f is a 1-1 mapping of X onto Y. Let q denote the intersection of Y_1 and Y_2 and let X_1, X_2, and p denote, respectively, $f^{-1}(Y_1)$, $f^{-1}(Y_2)$, and $f^{-1}(q)$. Then $Y_1 - q$ and $Y_2 - q$ are disjoint open sets in Y. Consequently, $X_1 - p$ and $X_2 - p$ are disjoint open sets in X. Since, X is connected, this implies that each of the sets X_1 and X_2 is connected. A similar argument shows that for each connected open neighborhood N of p in X, each of the sets $N \cap X_1$ and $N \cap X_2$ is connected. Thus, each of X_1 and X_2 is locally connected at p and, therefore, locally connected. Since each of the sets Y_1 and Y_2 is closed in Y, each of X_1 and X_2 is a closed subset of the locally compact space X and is therefore locally compact. Because each of Y_1 and Y_2 has Property H, it now follows that $f \mid X_1$ is a homeomorphism of X_1 onto Y_1 and $f \mid X_2$ is a homeomorphism of X_2 onto Y_2. Consequently, f is a homeomorphism of X onto Y.

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